Opportunity Cost of Capital for Entrepreneurs: a Reappraisal*

Prabesh Luitel†, Piet Sercu‡, and Tom Vinaimont§

January 5, 2024

Abstract

For undiversified entrepreneurs, the Cost of Capital (CC) is often set so as to align the project's Sharpe Ratio (SR) with the market's. Inserting annualised weekly-return volatilities from newly-listed firms—100%, on average—Kerins, Smith and Smith (2004) obtain a typical CC of 57%. We show that portfolio theory requires CCs substantially above the SR-based ones but we also argue that 100% volatilities are excessive: purely income-oriented entrepreneurs may even be happy with CCs below 20%. We provide closed-form return-on-value CCs for complete and partial non-diversification, as well as genuine pricing formulas based on dollar cashflow moments.

Keywords privately-held assets; undiversified investor; required return; committed owner; total-risk beta

JEL classification G31, G32

Acknowledgements Tom Vinaimont acknowledges a Social Policy Grant from Nazarbayev University.

Declaration of interest The authors have no material interest whatsoever linked to the issue discussed here.

Data; ethics We use Orbis accounting and Refinitiv stock-price data, for which a license is required. The article does not rely on tests on living beings.

Author contribution Authors are listed alphabetically. All authors contributed to the ideas, test design, and write-up.

Words The abstract has 98 words, by Microsoft's word counter, and the rest of the paper about 15,000, including math, numbers in tables, and references as counted by ASPOSE.

---

*We thank Frans de Roon for helpful discussions, Geert Dhaene for advice on autocorrelations, Onur Polat and Elisa Pazaj for comments at conferences in Penn State and Lille, and many participants in workshops in our own departments. Frans de Roon and Joy van der Veer (2023) independently derive much of our Proposition 2. We remain responsible for any remaining shortcomings.

†IESEG Paris–Lille; p.luitel@ieseg.fr; +33 155911010 ext 1162

‡KU Leuven; piet.sercu@kuleuven.be; +32 16 326 756

§Nazarbayev University, Astana; tom.vinaimont@nu.edu.kz; +7 7172 70 62 90

Contents

1 A valuation framework with non-traded assets **4**

1.1 The flaw in SR-based valuation 5

1.2 Portfolio choice when one position is not scalable 9

2 The committed investor's CC **12**

2.1 The single-asset required return 12

2.2 Numerical illustrations 13

3 The partially committed investor's CC **16**

3.1 The CC with constrained optimal portfolios 16

3.2 Numerical illustrations 18

4 The effects of model and parameter choices **22**

4.1 CCs based on returns-on-value inputs 22

4.2 Cashflow-based CCs (the income approach) 26

4.2.1 PV-ed income distributions with partially persistent shocks 26

4.2.2 Data and general procedure 28

4.2.3 Results 31

5 Discussion and conclusions **35**

References **38**

Appendices **39**

Privately-owned firms should be both rare and extraordinarily profitable, on average. That, at least, would be implied by the calculations of Kerins, Smith and Smith (2004), henceforth KSS. SME owners tend to be severely undiversified, and this, KSS infer, inflates the average entrepreneurs' cost of capital (CC)[1] all the way up to 57 percent, five times the average CC based on diversified ownership (the CAPM).

KSS do add that the 57-percent CC does not apply everywhere: if investors are merely underdiversified rather than totally undiversified, the reported CC drops to somewhat more modest levels, like 45 percent if the own firm represents one third of invested wealth. But even that hurdle looks hard to meet. Moskowitz and Vissing-Jorgensen (2002) find that private equity returns do not beat those on listed stocks, despite the lack of diversification. Cochrane (2005) questions the Moskowitz and Vissing-Jorgensen (2002) sample and discusses strong biases in estimates of returns and risks for other nonlisted assets, but in the end again comes up with a performance for non-listed firms that is not very different from that of listed stocks. Driesen, Lin and Phalippou (2012) similarly address biases in data and find no outperformance. While the private-asset record has improved since the financial crisis (Kartashova, 2014), the returns remain far below KSS' norms. As potential reasons why entrepreneurs would still invest, Moskowitz and Vissing-Jorgensen invoke strong skewness-preferences, non-pecuniary benefits (independence, autonomy, flexibility and so on), and overoptimism.[2] We do not dispute these. What we question is the model and the inputs behind the high hurdle rates.

Consider the risk inputs first. For the undiversified investor, KSS employ annualised standard deviations of 90-210 percent *per annum*, obtained from weekly returns on recently listed firms. Such a sample must contain a good dose of illiquidity and pricing noise (largely transient, but treated as permanent in the annualisation). More fundamentally, the procedure

[1] A table with definitions of symbols, variables and abbreviations is provided in the Appendix.

[2] For non-pecuniary dividends, see also *e.g.* Hickman, Barnes and Byrd, 1995; Xu and Rueff, 2004; Benz and Frey, 2008; Hurst and Pugsley, 2011; Hvide and Panos, 2014. For optimism, see Landier and Thesmar, 2009.

equates risk with the uncertainties about what the firm could fetch at a future time T if it had been listed. This looks logical for venture capitalists preparing for an IPO, but in many other cases this does not apply, notably when the owner of an unlisted small business expects to stay in the firm for a long time. If so, they care mostly about cash-flow and its implication for subsequent years (the 'income-oriented' view), not so much about a nearby exit value.

Also the 'total risk' or 'total beta' CC standard behind the 57-100% CC deserves reconsideration. In that view, portfolios are priced via their Sharpe-Ratio (SR).[3] On reflection, though, SR-based pricing does not mesh well with a key feature of an entrepreneurial investment, its lack of scalability. *Ex ante*, the entrepreneur may consider two or three alternative versions, but once the best variant is selected, the project is all or nothing. The investor cannot decide to shrink the selected version to, say, 1% of the proposed size and still hope to receive 1% of the originally forecasted cashflows. Nor can they easily sell 99% of their firm to outside shareholders and use the proceeds to diversify, as portfolio theorists would recommend: for tiny firms with zero reputation, information costs or adverse-selection discounts would make such a placement quite expensive, and the owner's non-monetary benefits complicate any dealing with potential financial co-investors even more. True, there are business angels and venture capitalists who may step in, but many small and unglamorous firms remain on their own.[4]

The all-or-nothing nature of the project matters. If all assets were tradable, a proposed

---

[3]To our knowledge, the use of the securities-market line for the valuation of an undiversified holding was first proposed in a journal by Hickman, Barnes and Byrd (1995), who refer to Pratt, Reilly and Schweihs (1981)'s book; it was later popularised by A. Damodaran in his 2002 and 2012 books. Butler and Pinkerton wrote many articles about what they now call 'the Butler-Pinkerton model' in trade publications *Valuation Strategies* or *Business Valuation Review.* There is little support in academia. Hickman, Barnes and Byrd (1995), Pereiro (2010), Hickman et al. (1995), Kerins *et al.* (2014) and Pattitoni and Savioli (2011) are among the few exceptions. Outside the mean-variance framework, Abudy, Benninga and Shust (2016) discuss a theoretical binomial model where the sole owner uses probabilities adjusted for their own consumption risk. De Roon and Van der Veer, in a draft book, independently reject SR-based pricing and derive results similar to our Proposition 2.

[4]Even in the U.S., privately-held firms dominate the economy. According to Asker, Farre-Mensa and Ljungqvist (2015), of the 5.7 million firms in the US in 2010 only 0.06 percent were publicly traded. Even among mid-size firms with over 500 employees, 86.4 percent were private. Relatedly, small-business owners are quite undiversified, as documented by e.g. Bitler et al. (2005), Kartashova (2014), Moskowitz and Vissing-Jorgensen (2002), or Mueller (2011). Boot, Gopalan and Thakor (2006) discuss the issues at stake.

portfolio that has a competitive Sharpe Ratio (SR) could always be re-mixed with risk-free assets and thus shifted up or down the security-market line to suit the owner's risk–return preferences. That is not an option when the portfolio is not tradable and not otherwise scalable. It follows that, almost surely, an extra return is required for moving the portfolio away from the entrepreneur's preferred point on the security-market line.

A further modeling issue is that, in KSS's calculations, any spare cash left after implementing the project is assumed to be fully invested in the market. We are rather ambivalent about this treatment of the partial-diversification case. On the one hand, it is not at all obvious how the entrepreneur would come up with a shadow distribution of the shop's possible market values that reflects all factors the market would have taken into account and can be integrated with the traded-asset distributions into an overall portfolio problem. Behavioralists, indeed, would even say that is not at all how real-world agents behave: under the mental-accounting view, the own firm is likely to be treated the same way as one's house, car, and kids' tuition fund: as an isolated investment, even if it is not really the owner's sole asset. On the other hand, if and when one nevertheless does adopt the integrated-portfolio view, then the KSS approach does not go far enough. Assuming that all spare cash is invested in the market index without any room for even risk-free lending or borrowing, the other key asset in portfolio theory, makes unstated and case-dependent assumptions about wealth and risk aversion. One should allow the investor more flexibility in their portfolio choice—still with the proviso that the non-traded firm has a non-flexible size in terms of future dollar income.

The two theoretical contributions of this paper accordingly are that we (i) amend SR-based valuation of non-scalable investments and (ii) explore the outcomes when the investor has flexibility in the remainder of their portfolio choice. We easily derive revised CCs in the familiar return-on-value form.[5] Since return-on-value CCs induce a self-referencing issue,

[5]By return on value, Fama and French (1999) refer to payoffs as a fraction of the current market price implied by the model—the very number the entrepreneur wants to identify.

our next contribution is (iii) to provide PVs and CCs written as functions of the expected value and risks of future dollar payoffs rather than of moments of returns on value. Our last contribution is empirical: the required return under the income view, inferred from over 360,000 non-listed firms as a lower bound on the CC. This delivers lower bounds to the CC that are well below 20 percent.

1 A valuation framework with non-traded assets

Consider an investor i with initial wealth $W_{i,0}$ who wants to value, in a portfolio-theory framework, a firm or project j. For simplicity the subscript j is omitted everywhere except for returns, where we need to distinguish between $\tilde{r}_j$ (the project), r_0 (the risk-free asset), $\tilde{r}_M$ (the market) *etc.* This project requires an investment I_0 and pays a time-1 income $\tilde{x}$ with a known dollar mean and variance. Multi-period problems are postponed until Section 4.2.1.

In the standard committed-investor case as discussed in the practitioners literature the tacit assumption is that $W_{i,0}$ equals I_0. One can easily add cases where $W_{i,0}$ is below I_0, in which case the shortfall is borrowed and the servicing of that debt is subtracted from the cashflows. KSS' partially-committed case, in contrast, postulates that, when there is a positive amount of cash left after the investment (that is, when $L_{i,0} := W_{i,0} - I_0 > 0$), this is invested in the market portfolio. In either case, by assumption $W_{i,0}$ and I_0 automatically determine the size of the loan or the market investment. That is, in KSS' view there is nothing to optimise except the discrete choice whether or not to adopt the project. The investor decides on the basis of the mean and variance of the combination of cashflows with debt or traded assets.

To be acceptable, that proposed combination should deliver enough value to compensate for the lack of diversification compared to the return offered by traded assets. In that context, we define the project's equity value, $V_{i,0}$, as a notional time-0 valuation that delivers exactly enough return, given the project's risk, to make the under- or un-diversified portfolio rank at par with i's best alternative portfolio of listed assets (henceforth referred to as i's fall-back portfolio). As the i subscript in $V_{i,0}$ indicates, this valuation is subjective: it depends on i's risk attitude. There is no public market where shares in the project can be bought or sold at that price; $V_{i,0}$'s main role is to be compared to the required investment, I_0. For the one-period project, the return $\tilde{r}_j$ is shorthand for the time-1 net cashflow (including terminal value) accruing to the shareholders, scaled by its model-consistent PV; that is, $\tilde{r}_j := \tilde{x}/V_{i,0} - 1$.[6] The uncertainty is summarised by the standard deviation and market correlation of $\tilde{x}$ or $\tilde{r}_j$. Below, we use the terms sigma and volatility as synonyms for standard deviation. Asset 0 is risk-free, and M refers to the (assumedly efficient) market portfolio.

1.1 The flaw in SR-based valuation

The alternative to holding the proposed portfolio including a lumpy non-traded risky asset is to buy an efficient portfolio of traded assets. When comparing two entire portfiolios, it is then often argued, the investor should look at total risk (like sigma or its square) and require the same Sharpe Ratio (SR) as the one offered by the market portfolio. This equal-SR condition immediately implies a CAPM with a modified beta:

$$\text{to ensure} \quad \frac{\mathrm{E}(\tilde{r}_j - r_0)}{\mathrm{std}(\tilde{r}_j)} = \frac{\mathrm{E}(\tilde{r}_M - r_0)}{\mathrm{std}(\tilde{r}_M)} \quad \text{we need} \quad \mathrm{E}(\tilde{r}_j - r_0) = \left[\frac{\mathrm{std}(\tilde{r}_j)}{\mathrm{std}(\tilde{r}_M)}\right] \mathrm{E}(\tilde{r}_M - r_0). \tag{1}$$

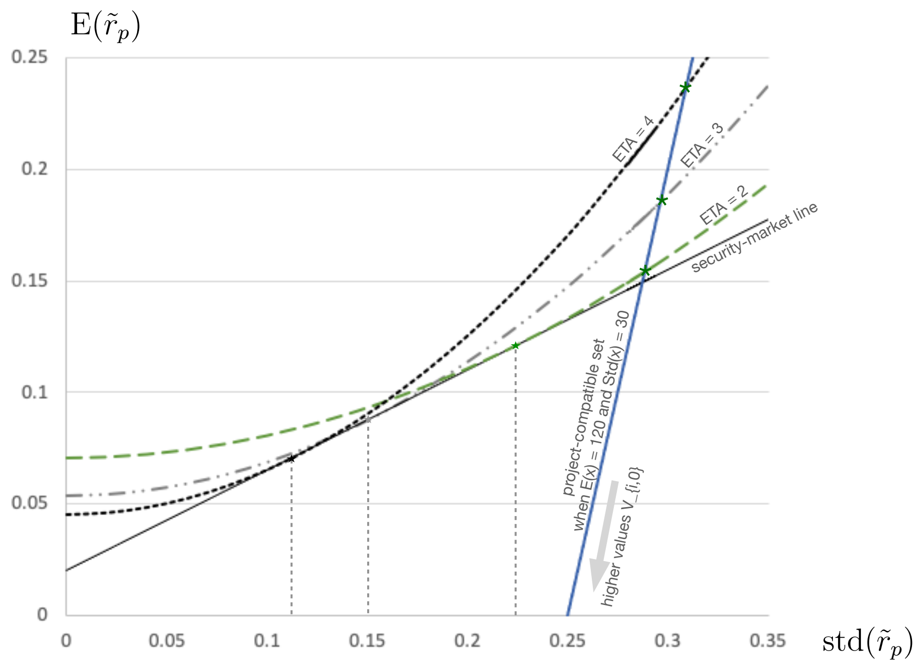
The square-bracketed fraction is often called the total-risk beta or total beta. The procedure can be summarised as follows:

SR investment rule *An investment is worthwhile if there exists a value $V_{i,0}$ that (i) implies the same expected return as the efficient portfolio with the same return volatility, and (ii) exceeds the required investment—i.e. the investment leaves* i *better off.*

This procedure is questionable because the SR criterion is borrowed, uncritically, from a

[6] The subjective nature of the PV scaler means that all $\tilde{r}_j$s should come with i subscripts, but we suppress those for the reader's convenience.

Figure 1: **Evaluation of a committed owner's project with** $\mathrm{E}(\tilde{x}) = 120$ **and** $\mathrm{std}(\tilde{x}) = 30$



Note We show the efficient set of traded asset as the full gray line, plus three illustrative indifference curves (for relative risk aversion (RRA) equal to 2, 3 or 4, shown as short-dash curve, the dash-and-dot curve, and the long-dash one), and the set of mathematically feasible risk–return combinations for a project whose cashflows have mean and sigma worth 120 and 30: $\mathrm{E}(\tilde{r}_j) = -1 + \frac{120}{30}\,\mathrm{std}(\tilde{r}_j)$.

framework where all assets are tradable and divisible. In that setting, one can always costlessly shift a proposed portfolio up or down the security-market line and thus select the point on the line that maximises utility. The typical committed owner/entrepreneur does not have that luxury, though, as we argued: the project is not scalable.

The capital-market line, depicting, the possibilities available via traded assets, will always be part of the solution to the valuation problem, but in the presence of inflexibility we need also the set of feasible risk–return combinations for the project as a whole (*i.e.* including either risk-free borrowing or, in KSS's view, a residual investment in the market, M). We call this the project-compatible set. With zero-flexibility portfolios like in KSS, this set is easily obtained by substituting $V_{i,0} = \mathrm{std}(\tilde{x})/\mathrm{std}(\tilde{r}_j)$ into $\mathrm{E}(\tilde{r}_j) = \mathrm{E}(\tilde{x})/V_{i,0} - 1$: the locus of project-compatible risk–return combinations for this project is

$$\mathrm{E}(\tilde{r}_j) \quad = \quad -1 + \frac{\mathrm{E}(\tilde{x})}{\mathrm{std}(\tilde{x})}\,\mathrm{std}(\tilde{r}_j). \tag{2}$$

This locus is shown in Figure 1 as the steep full line that cuts through the security-market line from below. Its intercept is at -1. In the graph we adopt the paper's running example: $r_0 = 0.02$, $\mathrm{std}(\tilde{r}_M) = 0.15$ and $\mathrm{E}(\tilde{r}_M - r_0) = 0.0675$ so that the market Sharpe Ratio (SR) equals $0.0675/0.15 = 0.45$; and, for the project, $\mathrm{E}(\tilde{x}) = 120$ and $\mathrm{std}(\tilde{x}) = 30$. (For use further down, note that the example's implied market average RRA is $0.0675/0.15^2 = 3$.) The slope of this project-compatible locus, then, is $120/30 = 4$. Each point on the halfline corresponds to a different $V_{i,0}$. A higher $V_{i,0}$, for given $\mathrm{E}(\tilde{x})$ and $\mathrm{std}(\tilde{x})$, corresponds to lowering $\mathrm{E}(\tilde{r}_j)$ and $\mathrm{std}(\tilde{r}_j)$, *i.e.* moving down- and leftward on the compatible-set line.

Note that the payoffs $\tilde{x}$ are explicitly those accruing to the shareholder, and as such they depend on both the payoff of the project *in se*, denoted below by $\tilde{y}$, and the investor's financial means. For a leveraged investment, the total expected payoff $\mathrm{E}(\tilde{y})$ might be 225 with a two-sigma confidence interval of 60 each side, and the required investment might be $I_0 = 200$. If the entrepreneur's wealth $W_{i,0}$ equals 100 and the missing 100 is borrowed at an r_D of 5 percent, the entrepreneur's expected income is $\mathrm{E}(\tilde{x}) = \mathrm{E}(\tilde{y}) - 100\,(1 + r_D) = 120$, which gives our feasible locus a slope of $120/30 = 4$. If $W_{i,0}$ is higher, say 150 or 200, the expected income after debt servicing would be up from 120 to 172.5 or 225, so that the locus' slope would rise from 4 to 5.75 or 7.50; that is, the feasible set would rotate up- or leftwards from its intercept at –1. Higher leverage, conversely, rotates the line right- and downward around its –1 intercept.

Recall that the value is identified by selecting the project-compatible risk–return combination that is at par with the traded-assets alternative. SR-based pricing equates that last condition to being on the security-market line. In our example, $V_{i,0} = 104.41$ meets those criteria: the excess-return and risk coordinates are $(120/104.41 – 1) – 0.02 = 0.129$ and $30/104.41$

= 0.287 with SR equal to 0.129/0.287 = 0.45. But, as we argued, being on the capital-market line equates to mean-variance efficiency only if the context is a traded-assets market, where one can move up or down the security market line at will. If i's choices are confined to the project-compatible set, efficiency is no longer sufficient for an investment prospect to be attractive. Figure 1 illustrates this. We show indifference curves for three entrepreneurs, whose relative risk aversions, denoted as η (ETA, in the graph), are equal to 4, 3 (like the market) or 2, respectively. None of these agents would be interested in the project when priced at 104.41. An SR-priced project is attractive only if i's preferred traded-assets portfolio (i's fall-back portfolio, henceforth generally denoted as p_i^*), happens to have exactly the same risk as the project. To leverage the market's sigma (0.15) up all the way to 0.287, such a p_i^* would have given the market portfolio M a weight of 0.287/0.15 = 1.913, an atypically high number that would reflect an RRA of 3/1.913 = 1.57, well below the market's RRA of 3.

There are more elements in the standard approach that deserve reconsideration. The typical practitioner's total-beta calculation takes for granted that $L_{i,0}$ equals zero, *i.e.* that $W_{i,0} = I_0$. If $L_{i,0}$ is negative, i can borrow. For better-off investors, KSS postulate that any positive $L_{i,0}$ is invested in the market, and the investor is assumed to value the project as part of their total portfolio. Behavioralists, as already argued, may disagree on the basis of a mental accounting argument and prefer an as-if-committed approach even when there are financial investments too. Orthodox finance scholars, on the other hand, would disagree with the KSS investment scenarios for the opposite reason: they do do not take the standard portfolio logic far enough. For the sake of generality, at the very least one ought to give i access to both the risk-free and market investments at the same time, regardless of what $L_{i,0} := W_{i,0} - I_0$ amounts to, and let this amount be allocated to traded and/or risk-free assets, in proportions chosen by i rather than fixed *a priori*.[7]

[7]For maximal generality, we should have given i access to all individual risky assets separately, but the predictable outcome would have been that i holds (i) the tangency portfolio, and (ii) a 'hedge' consisting of a negative position in a portfolio that optimally tracks the project in the sense of maximal R^2—see Adler and Detemple (1988), if necessary. It is plausible that the correlation between j and the other assets mostly

1.2 Portfolio choice when one position is not scalable

The non-traded asset has a subjective value $V_{i,0}$ in the sense defined before, and that value does include the investment's NPV. We denote that NPV, $V_{i,0} - I_0$, by $\Delta_{i,0}$. The investor accepts the valuation as fair if the implied expected return is in line with the implied return sigma, in the following sense:

Amended investment rule *An investment is worthwhile if there exists a value $V_{i,0}$ that (i) makes the un(der)diversified portfolio rank at par, in terms of utility from expected return and risk, with the fall-back portfolio p_i^* that* i *would have chosen otherwise, and (ii) exceeds the required investment amount—i.e. adoption makes* i *better off.*

In Figure 1, this looks easy enough: one selects the project-compatible point that is on the same indifference curve as the fall-back portfolio p_i^*. But such a project-compatible set is defined only for portfolios whose weights are set mechanically: borrow when $L_{i,0}$ is negative, invest in the market portfolio if $L_{i,0}$ is positive. This approach we want to avoid until we have verified whether and when those pre-set portfolio strategies are optimal.

The project's payoff, prior to adding debt or market investments, is $\tilde{y}$ dollars.[8] Let $\omega_{i,M}$ and $\omega_{i,j}$ denote the weights assigned to the market portfolio M and the project j, respectively, in the portfolio that does include the project, while $w_{i,M}$ is used for M's weight in the fall-back portfolio p_i^*. If the project is accepted, the investor's wealth rises from $W_{i,0}$ to $W_{i,0} + \Delta_{i,0}$, which can be written as $(W_{i,0} - I_0) + (I_0 + \Delta_{i,0}) = L_{i,0} + V_{i,0}$. Below, we start from the portfolio return, immediately using the constraint that the ω for the risk-free asset equals $1 - \omega_{i,M} - \omega_{i,j}$. Next, since $V_{i,0}$ has to be identified endogenously, the project's weight and

reflects market-wide factors, in which case M can act as an approximate hedge. A more sophisticated answer requires information that a real-world i does not reliably have, as KSS note.

[8] There is no borrowing by the firm—without loss of generality: ignoring default issues, any corporate borrowing would generate an offsetting extra risk-free investments in the remainder of the shareholder's portfolio.

return are rewritten in terms of $V_{i,0}$. Line three re-arranges:

$$\begin{aligned}\tilde{r}_p &= r_0 + \omega_{i,M}(\tilde{r}_M - r_0) + \omega_{i,j}(\tilde{r}_j - r_0),\\ &= r_0 + \omega_{i,M}(\tilde{r}_M - r_0) + \frac{V_{i,0}}{V_{i,0}+L_{i,0}}\left(\frac{\tilde{y}}{V_{i,0}} - 1 - r_0\right),\\ &= r_0 + \omega_{i,M}(\tilde{r}_M - r_0) + \frac{\tilde{y} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}},\end{aligned} \tag{3}$$

The mean-variance maximand, $\mathrm{E}(\tilde{r}_p - r_0) - \frac{\eta_i}{2}\mathrm{var}(\tilde{r}_p)$, therefore takes the form

$$\omega_{i,M}\mathrm{E}(\tilde{r}_M - r_0) + \frac{\mathrm{E}(\tilde{y}) - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}} - \frac{\eta_i}{2}\left[\omega_{i,M}^2\mathrm{var}(\tilde{r}_M) + 2\frac{\omega_{i,M}\,\mathrm{cov}(\tilde{y},\tilde{r}_M)}{V_{i,0}+L_{i,0}} + \frac{\mathrm{var}(\tilde{y})}{(V_{i,0}+L_{i,0})^2}\right]. \tag{4}$$

The decision variable is $\omega_{i,M}$, and the associated first-order condition readily leads to the optimal weight for asset M:

$$\begin{aligned}\text{optimality: } \mathrm{E}(\tilde{r}_M - r_0) &= \frac{\eta_i}{2}\left[2\,\omega_{i,M}\mathrm{var}(\tilde{r}_M) + 2\frac{\mathrm{cov}(\tilde{y},\tilde{r}_M)}{V_{i,0}+L_{i,0}}\right],\\ \Rightarrow\quad \omega_{i,M} &= \frac{1}{\eta_i}\frac{\mathrm{E}(\tilde{r}_M - r_0)}{\mathrm{var}(\tilde{r}_M)} - \frac{1}{V_{i,0}+L_{i,0}}\frac{\mathrm{cov}(\tilde{y},\tilde{r}_M)}{\mathrm{var}(\tilde{r}_M)},\\ &= \underbrace{\frac{1}{\eta_i}\frac{\mathrm{E}(\tilde{r}_M - r_0)}{\mathrm{var}(\tilde{r}_M)}}_{=\text{fall-back } w_{i,M}} - \underbrace{\frac{V_{i,0}}{V_{i,0}+L_{i,0}}}_{=\,\omega_{i,j}}\underbrace{\frac{\mathrm{cov}(\tilde{r}_j,\tilde{r}_M)}{\mathrm{var}(\tilde{r}_M)}}_{=\,\beta_j},\end{aligned} \tag{5}$$

The first term on the right, $w_{i,M}$, is the weight that i assigns to the market, M, in their fall-back portfolio p_i^*. If entrepreneurs have lower RRAs than the general investor,[9] the fall-back market weight $w_{i,M}$ exceeds unity and induces borrowing. We also recognise $\omega_{i,j}$, which is j's weight in the optimal portfolio including the project, multiplied by β_j, the standard market-model beta from $\tilde{r}_j = \alpha_j + \beta_j\,\tilde{r}_M + \tilde{\epsilon}_j$. The item $-\omega_j\beta_j$ is the hedge part: for instance, if the non-traded asset, including its NPV Δ, takes up 50% of the portfolio and has a beta

[9]Risk tolerance among entrepreneurs has been studied by *e.g.* Xu and Rueff, 2004; Holm, Opper and Nee, 2013; Koudstraal, Sloof and Van Praag, 2016.

of 1.4, the hedge's weight amount to $0.5 \times 1.4 = 0.7$: combining the project with a short position in M worth 70% of the portfolio removes all market-related uncertainty from the project. This 'short' position is implemented as a subtraction from the fall-back weight for M, $w_{i,M}$. If $\eta_i = 2$ while $\eta_M = 3$, i's risk tolerance is $3/2 = 1.5$ times the market's, and so is therefore i's fall-back weight assigned to the index portfolio. From this fall-back $w_{i,M}$ of 1.5, 0.7 is removed to hedge the project. The resulting $\omega_{i,M} = 0.8$ and the assumed $\omega_{i,j} = 0.5$ then imply a risk-free position of -0.3.

We can now evaluate the two scenarios postulated in the prior literature, starting with the partial-commitment scenario. When acting optimally, the better-off investor with $W_{i,0} > I_0$ does not mechanically replace the fall-back portfolio's market investment of size $W_{i,0}\, w_{i,M}$ by a position of size $L_{i,0}$, the spare cash. Rather, the best strategy is to hold on to the fall-back portfolio, add the project and its best hedge, and fund the entire combination by their own $W_{i,0}$ and possibly borrowing. The net remaining weight will rarely amount to exactly $1 - \omega_{i,j}$, the mechanical weighting rule postulated in SR-based pricing.

Now consider, instead, the conventional committed-investor case with, by assumption, no position in M, as discussed above. That might be approximately optimal indeed, like when $w_{i,M}$, β_j and $\omega_{i,j}$ are all equal to unity, but situations like that cannot be assumed to hold in general. That said, the solution without M can be justified on more pragmatic grounds too. As argued above, when $L_{i,0}$ is low or negative and RRA is below the market's level, the optimal portfolio requires borrowing to obtain the optimal M position. Real-world loan markets, however, may be too imperfect to make this realistic—imagine a banker's reaction if a levered starter would ask for an extra loan to finance their optimal stock-market investment. Mental accounting provides a second argument to question the optimal-portfolio approach, as we saw. Thus, in what follows we do consider each approach separately, using the term 'committed investor' for both the single-asset investor and the mental-accounting agent, and the term 'partial commitment' for the integrated/optimised version presented above. Under both views we always go for equal utility as our norm, not equal SRs; that is, we seek the CC above the security-market line, not on it.

2 The committed investor's CC

2.1 The single-asset required return

In the committed-investor scenario as defined above, the feasible risk–return combinations are summarised by the project-compatible set introduced before. When $L_{i,0}$ is negative rather than zero and i accordingly has to borrow, we focus on funds accruing to the shareholder ($\tilde{x}$), meaning that the results are about the cost of equity (CE), not the cost of capital (CC) that would apply to an unleveraged project with parameters $\mathrm{E}(\tilde{y})$ and $\mathrm{std}(\tilde{y})$. If borrowing entails a tax gain and/or a cost of distress, those should be included in $\tilde{x}$.

The relevant $V_{i,0}$ must be (i) project-compatible and (ii) place the project on the same indifference curve as p_i^*, *i.e.* almost surely above the security-market line. Denote i's RRA by η_i and the market's RRA by η_M. Then, as demonstrated in the Appendix,

Proposition 1 *For the 'committed' investor, the valuation $V_{i,0}$ that makes project* j *rank at par, utility-wise, with p_i^* has the following characteristics:*
(a) The return 'on value', $\tilde{x}/V_{i,0} - 1$, satisfies[10]

$$\mathrm{E}(\tilde{r}_j) \quad = \quad r_{i,f} + \frac{1}{2}\frac{\mathrm{var}(\tilde{r}_j)}{\mathrm{var}(\tilde{r}_{p_i^*})}\mathrm{E}(\tilde{r}_{p_i^*} - r_0), \tag{6}$$

where $r_{i,f}$, the intercept of i's fall-back indifference curve, equals

$$r_{i,f} := r_0 + \frac{w_{i,M}}{2}\mathrm{E}(\tilde{r}_M - r_0) = r_0 + \frac{1}{2}\mathrm{E}(\tilde{r}_{p_i^*} - r_0). \tag{7}$$

[10] Alternative ways of writing this are discussed further down, when we compare with the partial-commitment solution.

(b) The private assessment $V_{i,0}$ that achieves this is

$$V_{i,0} = \frac{\mathrm{E}(\tilde{x})}{1 + r_{i,f}} \times \frac{1 + \sqrt{1 - 2\,(1 + r_{i,f})\,\eta_i\,\frac{\mathrm{var}(\tilde{x})}{[\mathrm{E}(\tilde{x})]^2}}}{2}. \quad \blacksquare \tag{8}$$

The return-on-value CC is quite different from the 'total-beta' CAPM. First, the risk-free benchmark is $r_{i,f}$, not r_0, a term that is independent of the project's risk but is still investor-specific. It corresponds to a counterfactual risk-free return that, to i, would have made a zero-risk portfolio as attractive as the fall-back p_i^*, as can be seen by considering a zero-risk project j in Equation (6). Also in the pricing version $r_{i,f}$ acts as a substitute r_0. Just compare the pricing in Equation. (8) to the valuation with partial commitment:

$$\text{[Eq. (19) with } L = 0\text{:]} \qquad V_{i,0} = \frac{\mathrm{E}(\tilde{z})}{1 + r_0}\,\frac{1 + \sqrt{1 - 2\,(1 + r_0)\,\eta_i\,\frac{\mathrm{var}(\tilde{z})}{[\mathrm{E}(\tilde{z})]^2}}}{2}.$$

The reason why the risk-free rate is not playing its usual benchmark role here is that, unlike in the partially committed case, i is not allowed to lend or borrow in an optimising way, meaning that there is no first-order condition that involves r_0. The second difference from the total-beta version is the project-specific part in the CC. We should use a ratio of variances not of sigmas; the ratio is to be divided by 2, the other half of the expected excess return being in $r_{i,f}$; and the premium for this measure of risk is the fall-back portfolio's excess return, not the market's.[11]

2.2 Numerical illustrations

Consider our earlier example, with $\mathrm{E}(\tilde{x}) = 120$, $\mathrm{std}(\tilde{x}) = 30$, $r_0 = 0.02$, and $SR_M = 0.45$. Assume that the potential committed owner selects a preferred diversified portfolio p_i^* with

[11] One can write the CC in terms of the market price of risk, but then $w_{i,M}$ shows up as part of the 'beta'.

Table 1: **Valuation of a simple project using three cost-of-equity (CE) formulas**

	PV	CE	std($\tilde{r}_j$)	SR
corrected single-asset CAPM (when RRA = 2.7272)	102.15	0.175	0.294	0.530
SR-based ('total-risk' CAPM, flawed)	104.41	0.149	0.287	0.450
standard CAPM for a traded project, assuming $\beta_j - 1$	111.03	0.081	0.270	0.225

Note In the first row, a one-period project is being valued by an undiversified owner such that its return is competitive with the best diversified portfolio the owner can get. The project's payoff has $\mathrm{E}(\tilde{x}) = 120$, and $\mathrm{std}(\tilde{x}) = 30$; the risk-free rate is 0.02 and the market's SR equals 0.45. Assume the potential committed owner selects a preferred diversified portfolio p_i^* with 110% invested in the market portfolio, so that $\mathrm{E}(\tilde{r}_{p_i^*} - r_0) = 0.07425$ and $\mathrm{std}(\tilde{r}_{p_i^*}) = 0.165$. The second row shows the relative-sigma counterparts, and the third row the standard-CAPM analysis (assuming a market correlation of 0.5, *i.e.* a unit beta).

a weight for M of $w_{i,M} = 1.1$. This means that $\mathrm{E}(\tilde{r}_{p_i^*} - r_0) = 1.10 \times 0.0675 = 0.07425$ and $\mathrm{std}(\tilde{r}_{p_i^*}) = 1.10 \times 0.15 = 0.165$, or equivalently an η_i equal to $\eta_M/1.1 = 3/1.1 = 2.727$. The PV then works out as $V_{i,0} = 102.15$. As shown in Table 1, the standard deviation and expectation of the return on value are both higher than what the total-beta model predicts, as is the Sharpe Ratio. We add, for completeness, the CAPM's answer (see Equation (11), below), assuming that the market correlation is 0.5 (*i.e.* the beta is about unity). The familiar effect of underdiversification is that the cost of equity (CE) exceeds what a diversified owner would require—more than twice, in this example. (This is not a reflection of irrationality; rather, it provides a lower bound to the costs that have stopped i from going public.) This also means that the required return is well above what the SR-based model suggests, the reason being that the SR rule would have been correct only if η_i equaled $3 \times 0.15/0.287 = 1.57$. Our investor, whose RRA equals 2.727, requires a 2.5% extra premium for abandoning their fall-back portfolio p_i^*.

For more general illustrations, the Appendix derives the following corollary:

Corollary 1 to Proposition 1 *The closed-form expression for the required return is*

$$\mathrm{E}(\tilde{r}_j) = \frac{\mathrm{E}(\tilde{x})}{V_{i,0}} - 1 = \frac{2\,(1+r_{i,f})}{1+\sqrt{1-2\,(1+r_{i,f})\,\eta_i\,\frac{\mathrm{var}(\tilde{x})}{[\mathrm{E}(\tilde{x})]^2}}} - 1, \tag{9}$$

$$= \frac{2\,(1+r_{i,f})}{1+\sqrt{1-2\,\frac{1+r_{i,f}}{\mathrm{std}(\tilde{r}_{p_i^*})}\,\frac{\mathrm{var}(\tilde{x})}{[\mathrm{E}(\tilde{x})]^2}\,SR_M}} - 1. \blacksquare \tag{10}$$

The committed owner's required return in Equation (10) is a convex function of the $\mathrm{std}(\tilde{x})/\mathrm{E}(\tilde{x})$ ratio, see Figure 2. This CE behaves rather differently, in its relation to the std/E ratio, to the (flawed) SR-based CE or to the CE implied by the CAPM. The latter can be inferred from the standard return-on-value equation, turned into the familiar pricing equation in line two, below, and then rewritten as an expected return. For the CAPM, this yields

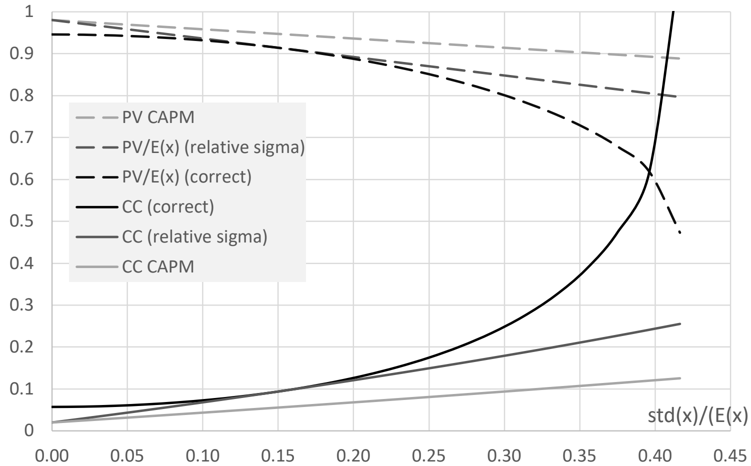
$$\begin{aligned}
\mathrm{E}\left(\frac{\tilde{x}}{V_{i,0}}-1\right)-r_0 &= \frac{\mathrm{cov}\left(\frac{\tilde{x}}{V_{i,0}},\tilde{r}_M\right)}{\mathrm{std}(\tilde{r}_M)^2}\mathrm{E}(\tilde{r}_M-r_0)=\mathrm{corr}\left(\frac{\tilde{x}}{V_{i,0}},\tilde{r}_M\right)\mathrm{std}(\tilde{x})\,\frac{\mathrm{E}(\tilde{r}_M-r_0)}{\mathrm{std}(\tilde{r}_M)};\\
\Rightarrow V_{i,0}^{capm} &= \frac{E(\tilde{x})-\mathrm{corr}\,(\tilde{x},\tilde{r}_M)\,\mathrm{std}(\tilde{x})\,SR_M}{1+r_0}; \qquad (11)\\
\Rightarrow \mathrm{E}(\tilde{r}_j)^{capm} &= \frac{\mathrm{E}(\tilde{x})}{V_{i,0}}-1=\frac{1+r_0}{1-\mathrm{corr}\,(\tilde{x},\tilde{r}_M)\,\frac{\mathrm{std}(\tilde{x})}{\mathrm{E}(\tilde{x})}\,SR_M}-1. \qquad (12)
\end{aligned}$$

Note that β_j can be written as $\mathrm{corr}(\tilde{r}_j,\tilde{r}_M)\frac{\mathrm{std}(\tilde{r}_j)}{\mathrm{std}(\tilde{r}_M)}$. It follows that the relative-sigma (or 'total-beta') CC can be obtained by setting $\mathrm{corr}()=1$:

$$\begin{aligned}
V_{i,0}^{SR} &= \frac{E(\tilde{x})-\mathrm{std}(\tilde{x})\,SR_M}{1+r_0}; \qquad (13)\\
\mathrm{E}(\tilde{r}_j)^{SR} &= \frac{\mathrm{E}(\tilde{x})}{V_{i,0}}-1=\frac{1+r_0}{1-\frac{\mathrm{std}(\tilde{x})}{\mathrm{E}(\tilde{x})}\,SR_M}-1. \qquad (14)
\end{aligned}$$

In the CAPM and relative-sigma CEs, written in terms of the dollar $\mathrm{std}(\tilde{x})/\mathrm{E}(\tilde{x})$ ratio, the curvature is hardly noticeable (Figure 2), but the single-asset CE rises rapidly with relative risk. Beyond $\mathrm{std}(\tilde{x})/\mathrm{E}(\tilde{x})=0.41644$, in our example, there is no solution. The technical reason is that the discriminant in the pricing equation (A.5) becomes negative. Economically, that means the project is too risky at any valuation. Recall that in Figure 1 a higher risk–return ratio means that the project-compatible set line rotates down- and rightward from its –1 intercept. So a negative discriminant corresponds to a situation where the project-compatible set is too far down to still cross i's fall-back indifference curve. One cause could be a low

Figure 2: **The CE and PV/E($\tilde{x}$) as a function of the dollar risk/return ratio**



Note A one-period project is being valued by an undiversified owner such that its return is competitive with the best diversified portfolio the owner can get. std($\tilde{x}$)/E($\tilde{x}$) varies from 0 to 0.40. The risk-free rate is 0.02 and the market's SR equals 0.45. Assume the potential committed owner selects a preferred diversified portfolio p_i^* with 110% invested in the market portfolio, so that $\mathrm{E}(\tilde{r}_{p_i^*} - r_0) = 0.07425$ and $\mathrm{std}(\tilde{r}_{p_i^*}) = 0.165$. Also shown are the CE and PV implied by the relative-sigma equation, and the CAPM's required return when the correlation between $\tilde{x}$ and $\tilde{r}_M$ is 0.5.

$W_{i,0}$ and, therefore, high leverage and high risk. So low wealth often prevents investments—a Matthew effect, as Robert K. Merton *sr*, the sociologist, coined outcomes where richer people face a superior opportunity set (Matthew 25:29).

This concludes our discussion of situations with $L_{i,0} \leq 0$ and $\omega_{i,M}$ fixed at zero. We now turn to cases where the own-firm investment is still constrained but $\omega_{i,M}$ is chosen optimally. While this may be irrealistic when $L_{i,0}$ is negative, it is less implausible when $L_{i,0} > 0$—provided the investors can easily map all sources of uncertainty into a (latent) value distribution for the project, modeled jointly with that of traded assets.

3 The partially committed investor's CC

3.1 The CC with constrained optimal portfolios

Recall that i's optimal way to invest in a private asset is to add it to the fall-back portfolio p_i^*, accompanied by the hedge, see Equation (5). The investor's best utility level then follows by substituting this optimal weight (5) into the utility function (4). It is convenient to first write, in Equation (3), the excess return in terms of beta-hedged cashflows, $\tilde{z}$:[12]

$$\begin{aligned}
\omega_{i,M}(\tilde{r}_M - r_0) + \frac{\tilde{y} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}} &= \left[w_{i,M} - \frac{\operatorname{cov}(\tilde{r}_j,\tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}\frac{V_{i,0}}{V_{i,0}+L_{i,0}}\right](\tilde{r}_M - r_0) + \frac{\tilde{y} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}}, \\
&= \left[w_{i,M} - \frac{\operatorname{cov}(\tilde{y},\tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)\,(V_{i,0}+L_{i,0})}\right](\tilde{r}_M - r_0) + \frac{\tilde{y} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}}, \\
&= w_{i,M}(\tilde{r}_M - r_0) + \frac{\tilde{y} - \frac{\operatorname{cov}(\tilde{y},\tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}(\tilde{r}_M - r_0) - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}}, \\
&=: w_{i,M}(\tilde{r}_M - r_0) + \frac{\tilde{z} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}}, \qquad (15) \\
\text{where } \tilde{z} &:= \tilde{y} - \frac{\operatorname{cov}(\tilde{y},\tilde{r}_M)}{\operatorname{var}(\tilde{r}_M)}(\tilde{r}_M - r_0). \qquad (16)
\end{aligned}$$

Thus, with optimal investment, expected utility equals

$$\mathrm{E}(\tilde{r}_p) - \frac{\eta_i}{2}\operatorname{var}(\tilde{r}_p) \text{ where } \tilde{r}_p = \begin{cases} r_0 + w_{i,M}(\tilde{r}_M - r_0) + \frac{\tilde{z} - V_{i,0}(1+r_0)}{V_{i,0}+L_{i,0}}, & \text{if } j \text{ is accepted,} \\ r_0 + w_{i,M}(\tilde{r}_M - r_0), & \text{if not.} \end{cases} \qquad (17)$$

The project-compatible halfline that helped identify the solution in the committed-investor case now is replaced by a locus of feasible risk-return combinations that is concave. Each point on the locus corresponds to a possible valuation $V_{i,0}$, and the higher the plugged-in number, the more southward the point on that locus. For the purpose of accepting/rejecting the project, the critical valuation is the highest $V_{i,0}$ that still makes the outcome rank at par with the fall-back solution p_i^*. As shown in the Appendix,

Proposition 2 *To an agent who can add optimal positions in the market portfolio and the risk-free asset, the valuation that makes the underdiversified portfolio rank at par, utility-wise,*

[12] Recapitulating, $\tilde{y}$ denotes the pure project's cashflow, $\tilde{x}$ is $\tilde{y}$ minus debt service (if any), and $\tilde{z}$ is $\tilde{y}$ hedged against r_M risk.

with p_i^* has the following characteristics:

(a) The expected return on value, $\tilde{r}_j := \tilde{y}/V_{i,0} - 1$ with market-model residual $\tilde{\epsilon}_j$, has an expected value of

$$\mathrm{E}(\tilde{r}_j - r_0) = \beta_j \, \mathrm{E}(\tilde{r}_M - r_0) + \frac{\omega_{i,j}}{2} \frac{\mathrm{var}(\tilde{\epsilon}_j)}{\mathrm{var}(\tilde{r}_{p_i^*})} \mathrm{E}(\tilde{r}_{p_i^*} - r_0). \tag{18}$$

(b) The private assessment $V_{i,0}$ that achieves this is

$$V_{i,0} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{19}$$

with $a = (1 + r_0)$; $b = -\mathrm{E}(\tilde{z}) + L_{i,0}(1 + r_0)$; $c = (\eta_i/2)\,\mathrm{var}(\tilde{z}) - L_{i,0}\mathrm{E}(\tilde{z})$;

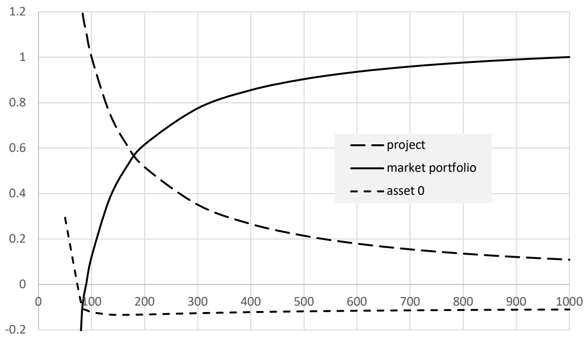
$$\mathrm{E}(\tilde{z}) = \mathrm{E}(\tilde{y}) - \frac{\mathrm{cov}(\tilde{y}, \tilde{r}_M)}{\mathrm{var}(\tilde{r}_M)} \mathrm{E}(\tilde{r}_M - r_0); \mathrm{var}(\tilde{z}) = \mathrm{var}(\tilde{y})[1 - \mathrm{corr}(\tilde{y}, \tilde{r}_M)^2]. \blacksquare$$

In the return-on-value Equation (18), the first term on the right represents the cost of hedging. For capital budgeting purposes, a reasonable-looking number for beta can be adopted, like in CAPM-based applications. The cost of hedging is topped up by a premium for underdiversification, which commonsensically shrinks the lower the project's value weight, the lower its residual risk in the market model, and the lower i's RRA as reflected in $\mathrm{E}(\tilde{r}_{p_i^*} - r_0)/\mathrm{var}(\tilde{r}_{p_i^*})$. Note that the effect of beta does not stop at the first term. A higher beta could be the result of a higher market correlation, a higher cashflow sigma, or a lower PV, and any these will also change the premium for hedged variance in convoluted, interacting ways.

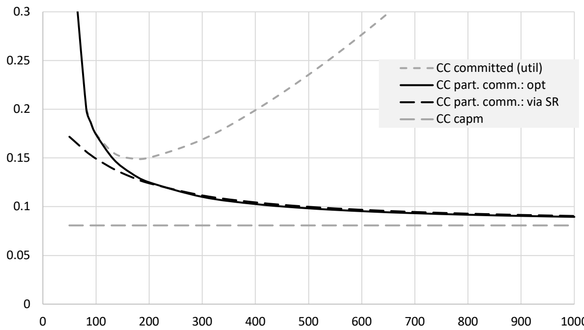
3.2 Numerical illustrations

In Panels A and B of Figure 3 we show that, at $L_{i,0} = 0$, the impact of allowing optimal market investments (with valuation still being utility-based, not SR-based) can be minimal. In that application we value the same project as before (*i.e.* $\mathrm{E}(\tilde{y}) = 120, \mathrm{std}(\tilde{y}) = 30, I_0 = 100$). We again set the correlation between $\tilde{y}$ and $\tilde{r}_M$ at 0.5, so that β_j is about unity, a reasonable

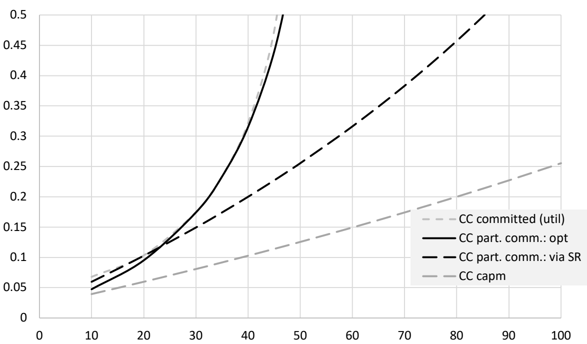
Figure 3: Portfolio weights and required return for the efficient investor



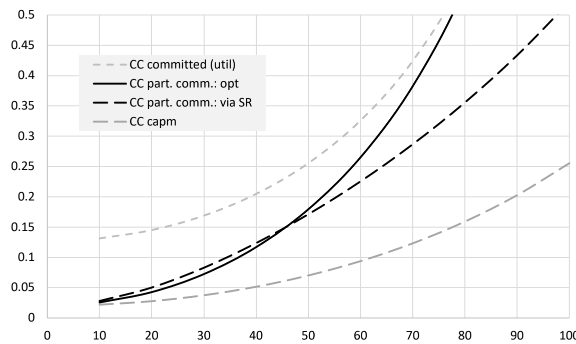
Panel A: Portfolio weights (M, j and 0) as a function of wealth



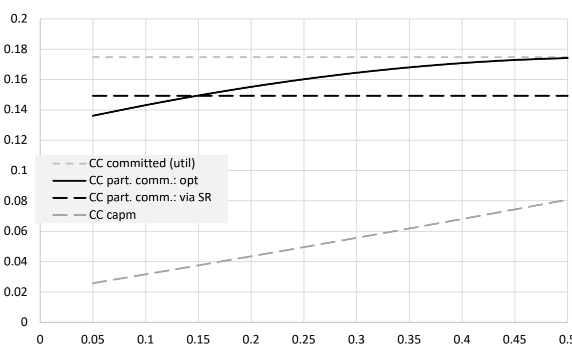
Panel B: CC as a function of wealth according to four models.



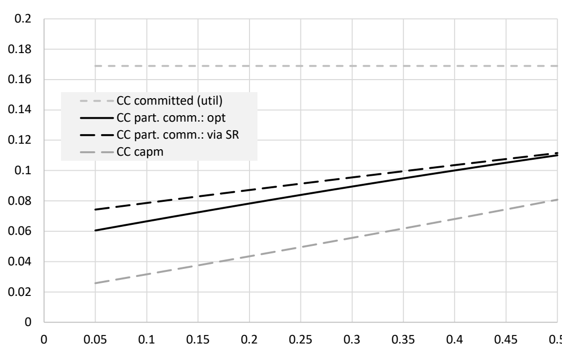
Panel C: CC (4 models) as a function of std(x); Wealth = 100



Panel D: CC (4 models) as a function of std(x); Wealth = 300



Panel E: CC (4 models) as a function of market correlation; Wealth = 100



Panel F: CC (4 models) as a function of market correlation; Wealth = 300

Note Panel A shows the optimal portfolio weights for an investor with RRA 2.72 (below the market's RRA, 3) who holds assets 0 (the risk-free bond, paying $r_0 = 2\%$), M (the market portfolio with $\mathrm{E}(\tilde{r}_M) = 6.75\%$ and $\mathrm{std}(\tilde{r}_M) = 30\%$) and an indivisible project j whose cashflow has mean 120, std 30, market correlation 0.5 and cost 100. Initial wealth varies between 100 ($= I_0$) and 1000. Panel B plots the corresponding CCs by four formulas, with the correct one shown as the full black curve. Panels C and D let the project std vary from 10 to 100, for wealth 100 (C) or 300 (D). In Panels E and F std is back at 30, and corr varies between 0.05 and 0.5, for wealth 100 (E) or 300 (F). The CC formulas include, at the lower end, the CAPM (irrelevant, but an interesting benchmark) and at the upper end the CC set by an investor who reluctantly places excess wealth in risk-free assets ('committed' (util)). The logically flawed 'partially committed' CC, based on SRs, and the optimal-investment variant take the middle ground.

level,[13] and we let i's initial wealth range from 100 to 1000. The case with wealth 100 is closest to the committed-investor case of Proposition 1, but not entirely identical. Given that (i) i's fall-back weight for M, $w_{i,M}$, equals 1.1, and (ii) the project has a unit portfolio weight and a unit beta, there is a small net investment in the market left, after hedging, and this small net position is financed by a loan. But the valuation is almost unaffected (102.20, up from 102.15 without investments in M), and so of course is the CC (0.1742 instead of 0.1747). This near-equality of outcomes holds for a wide range of RRAs, as we shall see in Figure 4 below; that is, given the sigma chosen here the crucial ingredients for the valuation are $L_{i,0} = 0$ and $\beta \approx 1$. The CC is far below KSS' because the assumed volatility is much lower.

The effects of higher values of wealth and $L_{i,0}$ can be read off, still in panels A and B of Figure 3, by moving to the right on the horizontal axis. When $W_{i,0} = 200$, *i.e.* when $L_{i,0} = 200 - 100 = 100$, the respective weights of M, j and asset 0 become 0.61, 0.51 and –0.12; the CC drops from 0.175 to 0.125; and the valuation rises to 106.66. Predictably, further increases in $W_{i,0}$ continue these trends. At $W_{i,0} = 1000$, the CC has dropped to 8.96%, not too far from the CAPM's 8.08%.[14]

Next, consider the predictions when i wants to borrow and actually pulls this off. One can numerically find that, for the case $I_0 = 100$, the efficient investor's NPV becomes zero (*i.e.* the CC is 0.20) when $W_{i,0} = 81.82$. Recall that such an efficient investor uses M to hedge, which here even means shorting M and investing the proceeds risk-free. In contrast, a 'committed' i worth 81.82 who does not take any M position would set the required return somewhat above

[13]The beta is reasonable for a traded asset, and therefore also for a value mentally constructed by an all-knowing, rational-expectations agent who takes into account all factors a market would have considered if the asset were traded, as assumed in the optimal-portfolio approach. The beta of a pure cashflow, as we shall see, is much lower.

[14]As a parenthesis, a hypothetical committed investor who, like in Section 2, never holds listed stock and invests $L_{i,0}$ risk-free instead, would react very differently. With this portfolio rule, taking up the project including the (by assumption, mandatory) risk-free investment of all left-over cash, would become increasingly unappealing the larger $W_{i,0}$ and would induce rapidly increasing extra required returns from the project to compensate for the low zero-risk yield. This just shows that, for $W_{i,0} > I_0$, it would make no sense to rule out market investments. Optimal portfolios for relatively risk-tolerant investors tend to assign negative weights to the risk-free asset, not positive ones.

0.20, making the NPV marginally negative to them. More meaningful divergences do arise for lower wealths, though. At $W_{i,0} = 50$, the efficient investor sets a high CC, almost 40%, but the committed counterpart has already dropped out entirely: without the option to hedge, the discriminant is negative, meaning the fall-back alternative p_i^* is always preferred.

We can also compare this CC to the (flawed) SR version. In Panel, B, the SR model seems to do well enough, despite its conceptual weaknesses, for investors whose wealth is at least twice I_0. That, however, turns out to be a lucky strike tied to the assumed sigma and market correlation. Panels C to F in the same Figure show how the CC reacts when $\text{std}(\tilde{x})$ is gradually increased (from 10 to 100, at corr 0.5), or the correlation (from 0.05 to 0.5, at std 30), each time for wealths of either 100 or 300. In each of these graphs we show two more CCs, next to the correct one and the SR version. The CAPM-based version is always the lowest. It is not directly relevant for valuation purposes but does provide a lower bound to the costs of going public that prevent an IPO. Among the three remaining CCs, the highest is provided by the scenario of the 'committed investor' who inefficiently invests all excess cash (if any) in risk-free assets. The full black curve, next, shows the CC associated with an optimal choice from an asset menu containing the project, the market and the risk-free asset. Compared to that, the SR-based (and therefore logically flawed) variant of the partially committed investor CC is sometimes too low, sometimes too high, as the black dashed line shows. The unwarranted lenience stems from the fact that the SR-based approach still seeks a solution on the capital market line, not above it; the unnecessary harshness reflects the imposition of a sub-optimal portfolio, thus precluding efficient hedging.

Note, though, that in most of the above we are looking at volatilities less than 30 percent, which is substantially below what KSS work with. In the next section we see how the CCs fare, depending on the view on risk: a high one inferred from newly listed stocks, or a low one obtained from cashflows.

4 The effects of model and parameter choices

We open this section with a numerical evaluation of the return-on-value formulas, Equation (6) for the fully commited case and Equation (18) for partially committed investors with various levels of $L_{i,0}$. In these calculations we use the return-on-value sigmas estimated by KSS for various sets of young traded firms, which allows us to contrast our CC formulas with theirs for the same inputs (Section 4.1). Upon observing that, in our model, the CCs remain implausibly high even when RRA is set as low as 2, we then investigate the effect of adopting an 'income' perspective on risk (Section 4.2), which can be read as a lower-bound view.

4.1 CCs based on returns-on-value inputs

In Table 2, the parameters and results from KSS (2004) are reported in the panels 'Data', 'CAPM CC', and 'SR-based CC (KSS)'. The data part shows std($\tilde{r}_j$)s, market correlations, and betas for various portfolios of young firms, defined as firms with less than six years of listed history. The standard deviation is an annualised version of a shorter-period sigma, probably a weekly return.[15] KSS' parameters are $r_0 = 0.04$, $\text{std}(\tilde{r}_M) = 0.162$, and $\text{E}(\tilde{r}_M - r_0) = 0.08$, implying an η_M of 3.048, close to our earlier $\eta_M = 3$. KSS' CAPM-based CC is first calculated the standard way, to represent the end-investor's perspective; the 'gross' version adds back the part creamed off by typical VC funds, to identify the return the latter have to realise to satisfy the end investor. The four columns in the middle reproduce KSS's SR-based CCs, for project weights 100, 35, 25 and 15 percent. The five columns to the right, lastly, display our revised CC-s computed using Equation (6) for the fully committed case ('fulcom') and (18) for partially committed investors, again with project weights 100, 35, 25 and 15 percent.[16]

[15] The words 'day', 'daily' and 'month' are not found in the text, but KSS do mention a data-availability lower limit of 30 weeks.

[16] In our calculations, it will be recalled, the fully committed investor does not hold any listed stocks; the 'underdiversified' investor does, even in the border case where $L_{i,0}$ equals zero. This is why we have two solutions for $L_{i,0} = 0$.

Table 2: **Results for CC (percent): KSS versus revised formula**

		data			CAPM CC, %		SR-based CC (KSS), %				Revised CC for $\eta_i = 2$, $\eta_M = 3$				
		std(r)	corr	beta	net	gross	100%	35%	25%	15%	fulcom	100%	35%	25%	15%
All	All Obs.	1.02	0.20	0.99	11.4	16.7	57.5	45.6	40.0	31.1	115.7	115.0	48.0	37.7	27.4
By industry	Biotechnology	1.04	0.15	0.78	9.8	14.7	60.2	47.4	41.2	31.2	119.9	117.5	47.8	37.1	26.3
	Broadcast and cable TV	0.87	0.24	1.04	11.7	17.2	46.1	36.2	31.8	25.2	86.9	86.6	38.3	30.9	23.5
	Communication equipment	1.20	0.22	1.32	13.8	19.7	70.7	57.9	51.6	40.7	156.3	158.3	64.9	50.5	36.1
	Communication services	1.04	0.24	1.25	13.2	19.1	57.6	46.4	41.2	32.8	119.9	121.3	51.6	40.8	30.1
	Computer networks	0.93	0.21	0.98	11.2	16.5	50.7	39.8	34.8	27.2	97.9	97.1	41.7	33.2	24.6
	Computer services	1.44	0.17	1.22	13.1	18.8	96.9	80.7	72.1	56.3	220.7	221.8	86.6	65.8	45.0
	Retail, mail order, internet	1.06	0.22	1.17	12.6	18.3	59.6	48.0	42.4	33.4	124.2	124.9	52.4	41.2	30.1
	Software	1.37	0.20	1.37	14.2	20.2	87.2	72.5	64.9	49.2	200.7	203.1	80.8	62.0	43.2
Listing age	Year of IPO	2.12	0.04	0.39	6.9	11.1	291.7	252.1	228.9	178.8	466.7	461.2	166.0	120.6	75.2
	1–5 years after IPO	1.04	0.23	1.11	12.7	18.4	58.0	46.6	41.2	32.6	119.9	120.2	50.4	39.7	29.0
Revenue, profits	No revenue	1.19	0.17	0.98	11.3	16.6	72.2	58.4	51.4	39.5	153.9	153.1	61.3	47.2	33.0
	Revenue, negative income	1.35	0.20	1.33	13.8	19.8	85.1	70.6	63.1	49.6	195.2	197.2	78.5	60.3	42.0
Payroll	0–25 employees	1.00	0.20	1.00	11.4	16.7	55.4	43.9	38.5	29.9	111.6	111.0	46.6	36.7	26.8
	26–100 employees	1.26	0.12	0.73	9.5	14.3	81.0	65.4	57.3	43.1	171.3	168.5	65.4	49.5	33.6
	Over 100 employees	1.28	0.15	0.98	11.2	16.5	80.7	65.8	58.1	44.6	176.5	175.7	69.2	52.8	36.4

Note The panels headed Data, CAPM CC, and SR-based CC(KSS) are copied from KSS (2004), and show std, market correlations, and betas for portfolios of young firms, defined as firms with less than six years of listed history. KSS' parameters are $r_0 = 0.04$, $\mathrm{std}(\tilde{r}_M) = 0.162$, and $\mathrm{E}(\tilde{r}_M - r_0) = 0.08$, implying an η_M of 3.048. The five columns to the right are CC-s computed using Equations (**??**) for the fully commited case ('fulcom') and (18) for four examples of partially committed investors, with project weights 100, 35, 25 and 15 percent.

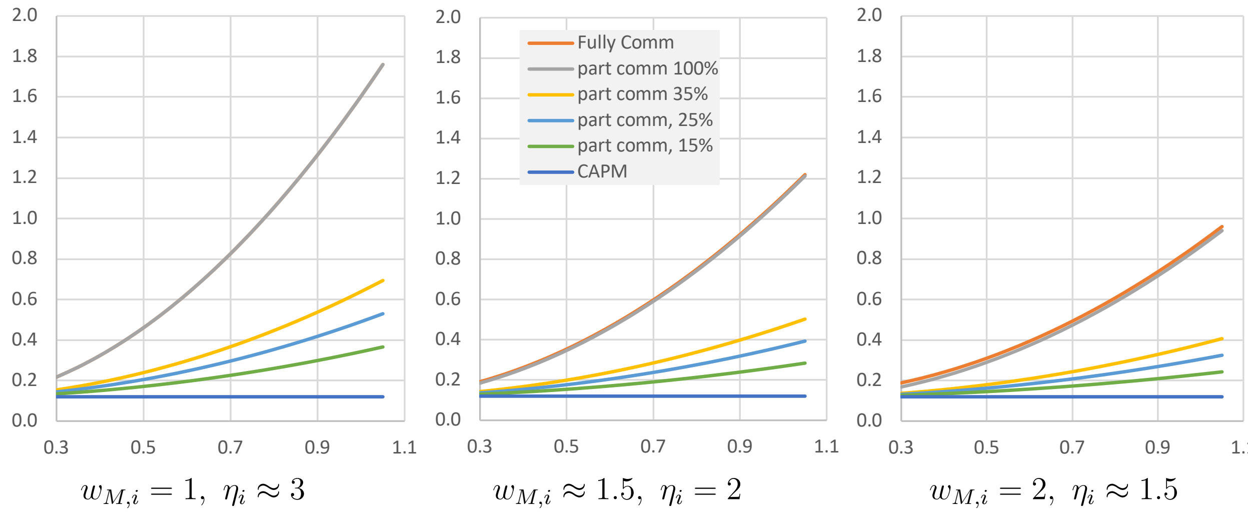
For these five CC-s the investor's fall-back portfolio's weight for M is set at 1.5, reflecting an RRA of $3.048/1.5 \approx 2$. Results for higher and lower RRAs follow later.

We first compare the revised CCs under the fully committed ('fulcom') scenario to those where market investments are possible but i has no spare cash—the 100% weight scenario. It turns out that, at $L_{i,0} = 0$ and $\beta_j \approx 1$, the option to invest in listed stocks has very little impact: even though i's RRA, at 2, is much below the market's, the CCs for the fully committed player are basically the same as for the underdiversified one with $L_{i,0} = 0$. (We already saw this in Panel B of Figure 3, but there the sigma was much lower and the risk-aversion higher.)

In both of the above zero-$L_{i,0}$ cases the CC is iso-utility based, so we now compare these hurdle rates to the (flawed) SR-based CCs. Two effects are competing: SR-based CCs are on, not above, the security-market line, and the SR looks at the total risk measures as a sigma, not the hedged risk measured as a variance. In Table 2 we see that, for investors with little or no spare cash, our CCs are about twice the (already hefty) numbers proposed by KSS. That effect is actually toned down already by our assumption that η_i equals about 2 not 3; with the latter RRA, the average CC for $L_{i,0} = 0$ rises to almost 180 percent, as we shall see. For $\eta_i \approx 2$, CCs do fall to KSS levels when the project weight is less than 0.50, and then end up below their standard for even milder degrees of underdiversification, situations where the option to chose $\omega_{i,M}$ and ω_0 freely becomes more important.

For many real-world starters underdiversification is severe rather than modest, and for them the high CCs proposed here—over 100 percent, when $L = 0$—would be the more relevant ones if one accepts the risk estimates. Many readers may agree that these CC standards are implausibly high. In Figure 4 we show similar results for a wider range of sigmas (on the horizontal axis), and provide those graphs for three different levals of RRA. The investor's fall-back portfolio's weight for M is either 1, 1.5 or 2, reflecting η_is of either 3.048 (leftmost graph), $3.048/1.5 = 2.033$ (middle) or $3.048/2 = 1.524$ (rightmost graph), respectively. In each graph, sigma varies from 30 to 105 percent. The CAPM CC is based on KSS's average

Figure 4: **Revised CCs for std($\tilde{r}_j$) from 0.30 to 1.05, and $w_{i,M}$-s of 1, 1.5, or 2**



Note The graphs show how various proposed CCs vary with return-on-value standard deviation, for a given level of RRA. KSS' parameters are $r_0 = 0.04$, std$(\tilde{r}_M) = 0.162$, and $\mathrm{E}(\tilde{r}_M - r_0) = 0.08$, implying an η_M of 3.048. We use the return-on-value formulas shown as Equations (**??**) for the fully commited case ('fullycom') and (18) for four examples of partially committed investors, with project weights 100, 35, 25 and 15 percent. The investor's fall-back portfolio's weight for M is either 1 (graph to the left), 1.5 (middle) or 2 (right), reflecting RRAs of 3.048, 3.048/1.5 and 3.048/2, respectively. The CAPM CC is based on KSS's average beta of unity.

beta of unity, and both r_0 and $\mathrm{E}(\tilde{r}_M - r_0)$ are set at 4 and 8 percent, as in KSS.

From the graphs, lowering the RRA is not the recommended way forward towards getting more reasonable CCs: a value of $\eta_i = 1.5$, implying a market investment $w_{i,M}$ of 200 percent in the fall-back portfolio, is already quite extreme and still proposes a CC of 90 percent for the case $L_{i,0} = 0$.[17]

The sigma has, unsurprisingly, a much more pronounced effect, and lowering that input should generate more plausible CCs. That would require a conceptual justification, of course, but we do think a case can be made that volatilities of 80-210 percent are too high. One reason for questioning such volatilities is that prices for very young listees must suffer from poor liquidity and lack of information, implying pricing errors that are, by their nature,

[17]Notice how in the graph on the right the two solutions for $L_{i,0} = 0$—without and with M positions—the CCs are different in a visually detectable way, unlike the cases with higher RRAs.

transient. If so, the negative autocorrelations for returns implied by transience are ignored when weekly returns are annualised in the conventional $\times\sqrt{52}$ way. Thin trading compounds this: too many returns are reported as zeroes followed by what is, in reality, a multiperiod return. It is hard to understand how, without pricing errors and thin trading, the average annualised sigma could be as high as 212 percent, KSS's estimate for the first year after the IPO. A second issue is whether the uncertainty about the market value at some unstated horizon, even after filtering out noise effects, is the relevant number at all. For VC- or serial-entrepreneur-style investors the capital gain or loss realised upon an exit via an IPO is what matters, but conventional long-horizon income-oriented entrepreneurs are different: they focus on cashflow, with the sales value as, at best, a background thought. In what follows we accordingly consider the extreme case where i takes into account only the risk of the cashflows, including their repercussions for the future.

4.2 Cashflow-based CCs (the income approach)

One can regard the above CCs as upper bounds, but such a bound is hardly informative: one does not need a model and data to justify the view that the CC is probably less than 100%. While observed weekly sigmas from tiny listed firms must overestimate the risks to i, the opposite holds under the strict income approach. The only risks taken into account are uncertainties associated with cashflows, including their repercussions for the future following, for instance, an ARIMA logic. The question then becomes whether the resulting lower bounds are more informative than the upper ones, for instance by being meaningfully above r_0. We first present our approach to multiperiod cashflow risk modeling (Section 4.2.1), then discuss the data (Section 4.2.2), and lastly proceed to numerical results (Section 4.2.3).

4.2.1 PV-ed income distributions with partially persistent shocks

The obvious difference between one- and multiperiod modeling is that the valuand, to be denoted by $\tilde{Y}_1$, includes not just the nearest one-period cash flow $\tilde{y}_1$, but also the time-1 PV of subsequent cashflows. The new issue, then, is how that future PV is updated in light of y_1. At one extreme, all cashflow shocks might be permanent, not a bad approximation for the large, listed firms studied by Gryglewicz, Mancini, Morellec, Schroth and Valta (2022). For small firms, where transient shocks are not mostly diversified away across many product or market lines, autocorrelation is likely to be lower than for the large firms in Gryglewicz et al. (2002). Early work on 'higgledy-piggledy growth' (Little, 1962; Reddaway, 1967, or Fuller and Levinson, 1992), plus our own empirical findings, tell us partial persistence dominates: of the unexpected cashflow relative to its expectation, a fraction ρ is repeated next period, a fraction ρ^2 the period thereafter, and so on. We also add expected growth at a rate g. Below we consider perpetuities, but finite-life versions are easily obtained.[18] Notationally compressing the required return $\mathrm{E}(\tilde{r}_j)$ into $\overline{r}$, we start from Y_1, the total value at time 1 conditional on the first-period realised cashflow, y_1. This conditional Y_1 consists of the (discounted) initial expectations about future cashflows plus the updates in those forecasts in light of the realised value of y_1:

$$\begin{aligned} Y_1 &= \mathrm{E}_0(\tilde{y}_1)\sum_{t=1}^{\infty}\left[\frac{1+g}{1+\overline{r}}\right]^{t-1} + [y_1 - \mathrm{E}_0(\tilde{y}_1)]\sum_{t=1}^{\infty}\left[\frac{\rho\,(1+g)}{1+\overline{r}}\right]^{t-1}, \\ &= \mathrm{E}_0(\tilde{y}_1)\,\frac{1+\overline{r}}{\overline{r}-g} + \big(y_1 - \mathrm{E}_0(\tilde{y}_1)\big)\frac{1+\overline{r}}{(1+\overline{r})-\rho\,(1+g)}. \end{aligned} \tag{20}$$

The mean and sigma to be used in the pricing equations of Propositions 1 and 2 then follow immediately:

$$\mathrm{E}_0(\tilde{Y}_1) = \mathrm{E}_0(\tilde{y}_1)\,\frac{1+\overline{r}}{\overline{r}-g} \quad \text{and} \quad \mathrm{std}_0(\tilde{Y}_1) = \mathrm{std}_0(\tilde{y}_1)\,\frac{1+\overline{r}}{(1+\overline{r})-\rho\,(1+g)}. \tag{21}$$

[18] Instead of the perpetuity expressions below, use $\sum_{t=0}^{n-1} f^t = (1-f^n)/(1-f)$ with f equal to $(1+g)/(1+\overline{r})$ or $\rho\,(1+g)/(1+\overline{r})$.

So the one-period risk, $\mathrm{std}_0(\tilde{y}_1)$, comes with a risk multiplier which starts from unity when ρ equals zero, and rises to the Gordon-Shapiro PE factor, $(1+\bar{r})/(\bar{r}-g)$, when ρ equals unity, the random walk. Growth modestly reinforces the effect of persistence. $\tilde{Y}_1$'s covariance with the market features the same risk multiplier, and so does the sigma of market-hedged cashflows.

Inevitably, in a multiperiod model, the mean and sigma in Equation (21) contain the CC as an argument, so to identify the solution we need an additional equation. This, of course, is that the PV produced by the propositions fits the predictions of the cashflow model:

$$V_{i,0} = \frac{\mathrm{E}_0(\tilde{Y}_1)}{1+\bar{r}} = \frac{\mathrm{E}_0(\tilde{y}_1)}{\bar{r}-g}. \tag{22}$$

So we are looking for a number $\bar{r}$ that reconciles Equations (21)-(22) to the valuation in Proposition 1 or 2.

4.2.2 Data and general procedure

We assess risk using two samples that are quite different in purpose. The first sample consists of 178 U.S. firms that have been in the Russell2000 index for 21 years and for which quarterly financial statements are available. This sample, downloaded from Eikon/Refinitiv, is biased in terms of the firms' size and survival. In empirical finance work, survival is obviously relevant because it biases the mean return, but here the effect on risk is equally important: an ongoing firm that has survived two decades is likely to be vastly more stable than the average starter. Still, one strength of the Russell2000 sample is that it comes with high-quality quarterly financial statements. In addition, it allows us to explore the link between accounting-return and market-return sigmas, opening the possibility of mapping accounting risk into return-on-value risk. This turns out to be an utterly unpromising approach. Unconditional cross-firm squared correlations between 21-year return-on-value sigmas and accounting-return sigmas as low as 0.10. Conditioning on size and industry, $\mathrm{std}(\tilde{r})$s can be predicted less imprecisely, but the relations seem unstable over time. Details are available on request. As return-on-value volatilities constructed from accounting returns are quite imprecise, in what follows we proceed directly from cashflow moments. That approach then raises a second issue with the Russell2000 data: in that sample, the cashflows behave more or less like a random walk, as also found for larger stocks by Gryglewicz, Mancini, Morellec, Schroth and Valta (2021). In the Orbis data on unlisted and smaller firms, the type we have in mind in this article, autocorrelations are much more diverse and lower.

The main sample, for the purpose of CC calculations, contains annual financial statements for a large set of nonlisted Eurozone firms,[19] obtained from the Orbis database on the basis of data availability (full data for 2013-2022). Compared to the Russell2000 set, most Orbis sample firms are much smaller, and the accounting information is low-frequency and probably less homogenous in quality. The ten-year-survival criterion may still bias the sample's risk and return, but less so than the Russell2000 sample (21 years). The main attraction is that these firms are not listed (like the SMEs we have in mind in this paper) and smaller. We download all data for Eurozone firms having full statements 2013-2022 in October 2023 (837,985 firms). We remove financial firms (15,721), companies with at least one negative entry for Total Assets (32,854), firms with zero standard deviations in cashflows (2,292) or with missing observations (65,852), or in the lower 1 percent of variance of the accounting return y/TA (8,284), all of which leaves us with 732,206 firms.

To render the parameters more comparable across firms with often very different sizes, all cashflows are rescaled to an initial Total-Assets base of 10 in year 1. Step 1 is to estimate all parameters firm by firm. Growth g is estimated via non-linear least-squares regression $y_t = a\,(1+g)^t$; the initial expected cashflow $\mathrm{E}_0(\tilde{y}_1)$ then follows as $\hat{a}\,(1+\hat{g})$, and the standard deviation as the regression's residual sigma. Since autocorrelation coefficients are from a

[19]The Eurozone restriction eliminates currency issues and most reporting standard heterogeneities that potentially plague international data sets.

Table 3: **Descriptives for the fize size groups in the Orbis sample**

Characteristics		total assets		sales	
size group	# firms	median	std	median	std
1	86,574	46	130	53	160
2	54,855	175	199	168	235
3	51,792	454	374	432	420
4	64,693	1,267	916	1,285	1,010
5	104,446	8,669	90,222	9,212	80,025

Parameters		$\mathrm{E}_0(\tilde{y}_1)/TA_0 \times 10$ (regression)		residual SD (regression)		growth (regression)		$\mathrm{corr}(\tilde{y}, \tilde{r}_M)$	
size group	ρ	median	std	median	std	median	std	median	std
1	0.405	1.876	4.959	2.086	7.800	0.027	0.617	-0.074	0.308
2	0.452	0.849	1.363	0.743	1.199	0.031	0.503	-0.065	0.304
3	0.532	0.640	0.978	0.497	0.778	0.047	0.436	-0.085	0.299
4	0.623	0.597	0.845	0.398	0.398	0.061	0.375	-0.114	0.292
5	0.688	0.615	0.771	0.329	0.564	0.058	0.347	-0.133	0.292

Note We sort over 700,000 EU non-listed firms with 10 years of data in Orbis into five size groups, each group being the diagonal cell in a 5×5, quintile-based sort on Total Assets and Sales. The table shows the number of firms in each diagonal cell, followed by median and std for total assets and sales, year-1 expected cashflow (the constant a in the non-linear regression $y_t = a\,(1+g)^t$, with y_t the cashflow rescaled by $10/TA_0$), growth (from the same regression), residual std (from the growth regression), and market correlation. The AR(1) ρ is corrected for small-sample bias following the procedure in Appendix A.2. Means and Min/Max observations are shown in Appendix Table A.3.

longitudinally small sample, they are severely biased, so we first deconvolute them following the procedure proposed by [the authors], here outlined in the appendix.[20] We then group firms by size, with size classes $s = 1, \cdots 5$ referring to firms that are in the sth quintile of both Total Assets and Sales (the diagonal cells in a two-way sort on Total Assets and Sales, that is). 362,360 firms are on the diagonal, which is almost half of the filtered sample. The size groups are constituted on the basis of the first year of data. For each group we compute medians for $\mathrm{std}(\tilde{y})$, g, $\mathrm{E}_0(\tilde{y}_1)$ and ρ from the constituent firms' parameter estimates, all displayed in Table 3, alongside sigmas. For the computations, the RRA levels are set at 2 or 3. The risk-free rate and market-return parameters are those used by KSS: $r_0 = 0.04$, $\mathrm{E}(\tilde{r}_M - r_0) = 0.08$,

[20]Autocorrelations follow the standard formula, allowing $\mathrm{std}(\tilde{y})$ to be different from $\mathrm{std}(\tilde{y}(-1))$, but the deconvolution procedure neutralises this. That is, the results are unaffected when we use the regression version of ρ.

$\text{std}(\tilde{r}_M) = 0.162$. Per size class we compute CCs for a syncretic firm whose characteristics each correspond to the group's median.

Some key characteristics for the five groups are shown in Table 3. Across the buckets, the top firms are more than 150 times larger, on average, than the small ones, whether we go by assets or sales. Expected initial cashflows relative to assets fall with the firm's size, but on the other hand large firms do grow faster. One-period sigmas fall with size. A counteracting effect is that ρ rises with size, but on balance the larger firms remain the lower-risk ones, as we shall see. In each quintile the estimated correlation with the market is slightly negative, on average, and more so the larger the firm. Lastly, for size classes 3-5 growth is above the risk-free rate. We return to those results later.

4.2.3 Results

Some issues arise when parameter inputs are estimates from historical data. First, in the partial-commitment calculations, risk premiums will turn out to be tiny. For a growth rate, being above the risk-free rate then also means exceeding the CC, which is incompatible with the perpetuity logic. As our objective is to obtain orders of magnitude for CCs, we relegate the results with the recently realised growth rates to the Appendix and, in the body of the text, discuss output where all gs have been set equal to 0.03, below the KSS 4 percent risk-free rate. Second, in the partial-commitment scenarios with substantial leverage ($L = -50$), there often is no solution: the implied risk-return combination obtained from the sample's estimated parameters is too far from the investor's reference indifference curve. A last issue is how to deal with median market covariance estimates which are negative in each of the size groups. Negative values for the population covariances are hard to understand, but do bear in mind that this is a longitudinally small sample with a huge common effect shared by all companies (the covid crisis). That is, we can hardly rely on cross-sectional diversification of errors, which makes the true sign of the typical betas less clear. For robustness, we therefore repeat the calculations assuming zero covariances. It makes little difference.

Table 4: **Purely income-based Costs of Capital, in percent, for fully and partially committed investors,** $g = 3\%$

	Cost of Capital, in percent								Present values, for initial TA = 10							
	full	partial commitment							full	partial commitment						
size group		L=-50	L=-5	L = 0	L=5	L=50	L=100	L=1000		L=-50	L=-5	L = 0	L=5	L=50	L=100	L=1000
covariances with $\tilde{r}_M$ **as estimated;** $\eta_i = 2$																
1	13.200	3.997	3.986	3.985	3.984	3.978	3.974	3.958	18.450	188.696	190.851	191.024	191.189	192.357	193.227	196.423
2	11.699	4.024	3.986	3.984	3.983	3.975	3.971	3.962	9.750	82.793	86.062	86.193	86.309	86.967	87.325	88.142
3	11.876	–	3.970	3.968	3.966	3.957	3.953	3.944	7.095	–	64.898	65.046	65.172	65.811	66.113	66.707
4	12.150	–	3.941	3.938	3.936	3.928	3.924	3.916	6.332	–	61.599	61.771	61.916	62.455	62.740	63.242
5	11.704	–	3.924	3.922	3.920	3.912	3.908	3.900	6.882	–	64.824	64.972	65.097	65.712	65.994	66.540
covariances with $\tilde{r}_M$ **as estimated;** $\eta_i = 3$																
1	11.104	4.023	4.004	4.003	4.001	3.991	3.985	3.961	23.224	183.886	187.429	187.702	187.962	189.815	191.123	195.896
2	9.447	4.074	3.999	3.996	3.994	3.983	3.977	3.963	13.157	78.990	84.933	85.132	85.309	86.300	86.835	88.056
3	9.647	–	3.985	3.982	3.979	3.964	3.958	3.945	9.474	–	63.935	64.161	64.352	65.313	65.763	66.652
4	9.997	–	3.957	3.953	3.950	3.935	3.928	3.917	8.281	–	60.535	60.798	61.019	61.980	62.421	63.196
5	9.552	–	3.936	3.933	3.930	3.917	3.912	3.901	9.142	–	63.979	64.197	64.382	65.293	65.713	66.480
covariances with $\tilde{r}_M$ **set zero;** $\eta_i = 2$																
1	13.200	4.053	4.038	4.037	4.036	4.029	4.024	4.006	18.450	178.779	181.269	181.460	181.641	182.922	183.862	187.158
2	11.699	4.081	4.028	4.027	4.025	4.016	4.011	4.002	9.750	78.437	82.474	82.620	82.751	83.459	83.924	84.660
3	11.876	–	4.032	4.030	4.027	4.016	4.011	4.002	7.095	–	60.993	61.156	61.293	61.968	62.277	62.852
4	12.150	–	4.040	4.035	4.032	4.018	4.012	4.000	6.332	–	55.717	56.006	56.164	56.912	57.238	57.943
5	11.704	–	4.035	4.032	4.029	4.017	4.012	4.002	6.882	–	57.880	58.054	58.200	58.902	59.214	59.783
covariances with $\tilde{r}_M$ **set zero;** $\eta_i = 3$																
1	11.104	4.085	4.059	4.057	4.055	4.044	4.036	4.008	23.224	173.476	177.745	178.031	178.303	180.244	181.672	186.640
2	9.447	4.160	4.044	4.041	4.038	4.025	4.018	4.003	13.157	73.149	81.248	81.477	81.677	82.769	83.335	84.581
3	9.647	–	4.050	4.045	4.040	4.024	4.017	4.003	9.474	–	59.999	60.247	60.552	61.470	61.932	62.814
4	9.997	–	4.058	4.053	4.048	4.027	4.019	4.003	8.281	–	54.746	55.037	55.278	56.403	56.890	57.785
5	9.552	–	4.055	4.049	4.045	4.025	4.017	4.003	9.142	–	56.803	57.088	57.322	58.414	58.885	59.728

Note CCs and values are calculated assuming that i is willing to ignore all sources of uncertainty other than the own time-1 cashflow. Data are ten years of cashflows and total assets, 2013-2022, on unlisted European firms, from Orbis. "Full" refers to full commitment: either all wealth is invested in the firm, or i at least treats it as a stand-alone asset. "Partial" commitment means the investor can borrow/lend and add optimal investments in the market index. L refers to the wealth left after the investment in the firm. We show CCs for firms that have 3% growth and, otherwise, the median characteristic in their size group. Such a 'size group' corresponds to the cell on the diagonal in a 5 by 5 double sort on the basis of total-assets and sales quintiles, 362,360 firms in total. When the estimated parameters mean the project cannot be attractive to the investor, no CC is shown.

Table 4 provides the key results for the CCs (in the left half of the table) and the $V_{i,0}$s (to the right). Inside each panel we show, per size or industry group, first a column with results for the fully committed investor, and then seven columns for partially committed investors, each with their own level of L, ranging from –50 to +1000. In interpreting these Ls, one can relate them to the $V_{i,0}$s shown in the rightmost half of the table, but their impact is small, here, as we shall see. Calculations are provided for RRAs of 2 and 3, in that order, and both sets are computed first for estimated market covariances and then for zero covs.

The CCs for fully committed investors in firms with the above characteristics range between 9.5 and 13.2 percent (or 8.2 and 13.8 percent when each group has its own g, see Appendix). More than half of the risk premiums stems from the project-independent part (*viz.* $r_{i,f} - r_0 = w_{i,M}/2\,\mathrm{E}(\tilde{r}_M - r_0)/2$, i's default indifference curve's intercept) which amounts to 4-6%.[21] The project's own risk then adds between 1.5 and 4.4%. All in all, the CCs are definitely not implausibly high. They are also meaningfully different from the risk-free rate, 4%, so in that sense this is an informative lower bound. That cannot be said about the partial-commitment CCs which, at least in the current income-based approach, basically returns the riskfree rate as the bound.

The differences between the full- and partial-commitment outcomes stem from the zero beta—directly, in part, but also via the implied return-on-value variances for each scenario. To explore the latter route, consider Table 5. In the table we first reproduce, per size group, the median $\mathrm{std}(\tilde{y}_1)$, g and ρ data, and we then infer the risk multiplier $k := (1+\bar{r})/[(1+\bar{r}) - \rho\,(1+g)]$ and $\mathrm{std}(\tilde{r}_j) = \mathrm{std}(\tilde{y}_i) \times k/V_{i,0}$, using the $V_{i,0}$ borrowed from Table 4 and shown next to k. We do so for both the fully and partially committed investor (with $L = 0$, but this choice hardly matters, with our numbers). The calculations are using CCs and $V_{i,0}$s from the top panel in Table 4: slightly negative betas, and RRA = 2.

[21]In the first term, $w_{i,M}/2$ equals 0.5 or 0.75 depending on η_i, and is multiplied by the market risk premium, 8%.

Table 5: **Return-on-value risk for full and partial commitment (at $L = 0$)**

	parameters			full commitment			partial commitment		
	std($\tilde{y}$)	g	ρ	multpl	V_0	std($\tilde{r}$)	multpl	V_0	std($\tilde{r}$)
1	2.086	0.030	0.405	1.583	18.450	17.90%	1.670	191.024	1.82%
2	0.743	0.030	0.452	1.714	9.750	13.06%	1.810	86.193	1.56%
3	0.497	0.030	0.532	1.960	7.095	13.72%	2.114	65.046	1.61%
4	0.398	0.030	0.623	2.339	6.332	14.70%	2.616	61.771	1.69%
5	0.329	0.030	0.688	2.735	6.882	13.07%	3.143	64.972	1.59%

Note $\tilde{y}$ is a cashflow rescaled to an initial Total Assets level of 10. From the parameters (std($\tilde{y}_1$), g and ρ) we infer the risk multiplier $k := (1+\overline{r})/[(1+\overline{r}) - \rho\,(1+g)]$ and, using $V_{i,0}$, the return-on-value risk $\text{std}(\tilde{y}_i) \times k/V_{i,0}$, once for the fully committed i and once for the partially committed investor with $L = 0$. The calculations are using the top panel in Table 4: slightly negative betas, and RRA = 2.

For the full-commitment i, implied return-on-value sigmas are 13-18%, comparable to the market's sigma and not manifestly unreasonable for agents willing to ignore all sources of value fluctuations other than income. That number drops to a counterintuitively tiny 1.6-1.8% in the partial-commitment calculations. Recall that these sigmas are calculated as $\text{std}(\tilde{y}_i) \times k/V_{i,0}$, with k the risk multiplier $(1+\overline{r})/[(1+\overline{r}) - \rho\,(1+g)]$. From the table, the risk multipliers are not very different between the full- and partial-commitment scenarios; that is, they are mostly driven by ρ, and less so by the CC. (They are, incidentally, not very high either, at 1.5-3.1: with $\rho \ll 1$ they remain far below the Gordon-Shapiro PE, which would be like 12-13 for CCs around 0.12-0.13 and g at 0.03 if ρ were 1.) The main impact on return volatility actually stems from the $1/V_{i,0}$ part: partial-commitment values, given the low CC, are roughly ten times higher, so they shrink the return sigmas by a factor 1/10 and the variance by a factor 1/100.

The low return-on-value variances explain some counterintuitive features of the output. First, except for one case (group 1, full commitment, $\eta_i = 3$), the return sigmas are below the market sigma (KSS' 16.2%). This means that, in the E()–std() plane the correctly priced projects are to the left of the market tangency point on the security market line. But in that domain the tangency indifference curve for $\eta_i = 2$ is above that for $\eta_i = 3$, which means that the more risk-tolerant investor actually demands a higher premium for moving away from their default portfolio. The low return-on-value variances also shed light on the inputs into the two CC equations, reproduced below in a way that facilitates comparison:

$$\begin{aligned}
\text{partial commitment:} \quad \mathrm{E}(\tilde{r}_j - r_0) &= \beta_j \; \mathrm{E}(\tilde{r}_M - r_0) + \frac{\omega_{i,j}}{2} \frac{\mathrm{var}(\tilde{\epsilon}_j)}{\mathrm{var}(\tilde{r}_{p_i^*})} \mathrm{E}(\tilde{r}_{p_i^*} - r_0), \\
&= \beta_j \; \mathrm{E}(\tilde{r}_M - r_0) + \omega_{i,j} \frac{\eta_i}{2} \mathrm{var}(\tilde{\epsilon}_j). \\
\text{full commitment:} \quad \mathrm{E}(\tilde{r}_j - r_0) &= \frac{w_{i,M}}{2} \mathrm{E}(\tilde{r}_M - r_0) + \frac{\eta_i}{2} \mathrm{var}(\tilde{r}_j). \qquad (23)
\end{aligned}$$

The low return-on-value sigmas tell us that, if one takes the income approach, the second term in the partial-diversification CC hardly matters:[22] In that model the premium for 'diversifiable' risk amounts to a few basis points. Ultimately, though, the difference between the two models' prediction is traced to the zero market covariance, which in itself wipes out the project-independent 4-6% part of the risk premium in the single-asset model, thus inflating the price, shrinking the return-on-value sigma, and further lowering the CC.

5 Discussion and conclusions

Kerins, Smith and Smith (2004) ask the pertinent question whether the lack of diversification we observe among many owners of SMEs has a large impact on the cost of capital (CC). Their calculations suggest the effect should be huge, with CCs like 57 percent on average and even up to 100 percent in some cases. That is puzzling: with CCs like this, why would entrepreneurs invest at all—unless much of the return is imagined, or comes as a non-pecuniary dividend from being independent etc. We do not argue with these explanations; our point is that the risk estimates and the CC model deserve a reassessment as well.

First we need to agree what the relevant choice is. There is a fall-back portfolio (p_i^*) of traded assets that the investor i would hold in the absence of the project. In one view ('the fully

[22] Given that β is (near-)zero, for current purposes we do not even need to distinguish between total risk or hedged risk.

committed investor'), the alternative to p_i^* is an investment in the firm and nothing but the firm (except, possibly, some borrowing). In the other view ('the partially committed investor'), the alternative to p_i^* is an optimal combination of the investment in the firm, traded assets, and lending or borrowing. The second view assumes sufficient financial means, unfussy lenders, and an unusual degree of sophistication for i, showing up in the fact that they even include their house(s), car(s), and appliances *etc* into their portfolio problem. In reality, the own firm is so different from traded assets—near-impossible to trade, for instance, and therefore with no independent and frequent information about its economic value—that integrating such an investment into an overall portfolio problem is easier said than done. Behavioral economists even question the very idea of an integrated evaluation ('mental accounting').

Consider, first, the fully committed investor's problem. The analysis should take into account that, typically, i's investment in the firm has an exogenous size, or at least that the optimal version of the project is a take-it-or-leave-it prospect rather than fully scalable. This means that one cannot travel up and down the capital-market line, adjusting the project's risk and return by re-mixing it with risk-free assets. In fact, the feasible $(\mathrm{E}(\tilde{r}_j), \mathrm{std}(\tilde{r}_j))$ pairs one can obtain by combining dollar cash-flow moments $(\mathrm{E}(\tilde{x}_j), \mathrm{std}(\tilde{x}_j))$ with a valuation $V_{i,0}$ are confined to a half-line $\mathrm{E}(\tilde{r}_j) = -1 + [\mathrm{E}(\tilde{x}_j)/\mathrm{std}(\tilde{x}_j)] \times \mathrm{std}(\tilde{r}_j)$. Given all this, the project is attractive compared to the fall-back portfolio p_i^* if it is (i) feasible in the sense just defined, (ii) μ/σ-wise located on the same indifference curve as p_i^*, and (iii) worth more than the required investment I_0. We provide a CC in the usual return-on-value form, but we also offer a genuine pricing equation that has dollar moments as its inputs. Being on the indifference curve through p_i^*, the required return is almost surely above the capital market line, not on it, as the 'total beta' procedure assumes. Using KSS' risk inputs, the CCs with this formula therefore look even more outlandish than theirs. The KSS return volatilities are probably excessive, though, being taken from newly listed firms whose volatile weekly returns suffer from illiquidity and lack of information and can, therefore, not be treated as fully permanent. Also thin trading (*i.e.* stale prices) inflates estimated sigmas.

If, at the other extreme, one adopts the 'income view' for valuation purposes, then the risk of cashflows (including their repercussions for subsequent incomes, if shocks are not fully transient) is what matters. This ignores many other factors: in principle, value depends not just on income but also on time-varying CC inputs, like the future risk-free rate and the market return's moments, and income is probably less well-behaved than our AR(1) process. For those reasons the 'income-based' valuation is definitely too optimistic. Pending an answer on how to model these missing parts, the income approach does produce informative lower bounds for CCs, in the 9.5-to-13.2 percent range (or 8.2-13.8 if each size group has its own g).[23] If true CCs are not too far above these bounds, one may need no non-pecuniary dividends, overoptimism, or skewness preference to understand why entrepreneurs do still set up their companies.

As implemented here, the income approach works less well in the partly committed investor's problem. The empirical fact is that cashflows in our (short) sample seem to be zero-beta, and this feature has strong, non-linear repercussions on the CC. In our data set, the resulting partially committed CCs are so low that they cease to be very informative even as a lower bound. To restore credibility, one would have to plug in a beta which would then reflect the non-income considerations behind value, the very factors the income approach is willing to ignore.[24] A unit beta, for instance, would lift the first term in the CC above its fully committed counterpart, $w_{i,M}/2\ \mathrm{E}(\tilde{r}_M - r_0)$. The resulting rise in the CC would then produce drastically lower $V_{i,0}$s and, therefore, return-on-value variances above those of the fully committed case. The effect of the higher variances on the CC is mitigated, of course, by a project weight that often would be below unity.

---

[23]In these calculations, the risk-free rate is 0.04, $\mathrm{E}(\tilde{r}_M - r_0) = 0.08$, and $\mathrm{std}(\tilde{r}_M) = 0.16.2$, like in KSS.

[24]To introduce a target beta, replace the estimated correlation by $\beta_j^* \times \mathrm{std}(\tilde{r}_M)/\mathrm{std}(\tilde{Y}_j) \times V_{i,0}$, with β_j^* denoting the postulated beta.

References

Abudy M, Benninga S., and Shust, E. 2016. The cost of equity for private firms. *Journal of Corporate Finance* 37, 431-443.

Adler, M. and Detemple, J.B., 1988, On the Optimal Hedge of a Nontraded Cash Position. *The Journal of Finance*, 43 143-153.

Asker, J., Farre-Mensa, J., Ljungqvist, A., 2015. Corporate investment and stock market listing: a puzzle? *Review of Financial Studies* 28, 342-390.

Boot , A.W.A., R, Gopalan and A.V. Thakor, 2006. The Entrepreneur's Choice between Private and Public Ownership. *The Journal of Finance* 61, 803-836.

Benz, M., and B. S. Frey, 2008. The value of doing what you like: Evidence from the self-employed in 23 countries. *Journal of Economic Behavior & Organization* 68(3-4), 445-455.

Bitler, M.P., Moskowitz, T.J., Vissing-Jorgensen, A., 2005. Testing agency theory with entrepreneur effort and wealth. *Journal of Finance* 60, 539-576.

Brennan, M. and A. Wang, 2010. The Mispricing Return Premium. *Review of Financial Studies* 23(9), 3437-3468.

Cochrane, J. H., 2005. The risk and return of venture capital. *Journal of Financial Economics*, 75(1), 3-52.

Damodaran, A. 2002 (2012). *Investment Valuation: Tools and Techniques for Determining the Value of Any Assets*, 2nd (3d) ed., 992. New York: 672–673 (in the third edition).

Damodaran, A, https://pages.stern.nyu.edu/ adamodar/podcasts/cfspr22/session10slides.pdf, p 187 (downloaded Oct 3, 2022).

De Roon, F. and J. van der Meer, 2023. Corporate Valuation and Discounting, draft e-book, https://valuation.kotobee.com/#/reader/chapter/0

Driessen, J., T. Lin, and L. Phalippou, 2012. A New Method to Estimate Risk and Return of Nontraded Assets from Cash Flows: The Case of Private Equity Funds. *Journal of Financial and Quantitative Analysis*, 47(3), 511-535.

Fama, E. F. and K. R. French, 1999. The Corporate Cost of Capital and the Return on Corporate Investment. *Journal of Finance* 54(7): 1939-1967.

Fuller, R. J., Huberts, L. C., and Levinson, M., 1992. It's not higgledy-piggledy growth!. *Journal of Portfolio Management*, 18(2), 38-52.

Gryglewicz, S., Mancini, L., Morellec, E., Schroth, E., and Valta, P. (2022). Understanding cashflow risk. *The Review of Financial Studies*, 35(8), 3922-3972.

Hickman, K. A., Barnes, C., and Byrd, J., 1995. A Note on Estimating the Cost of Capital for the Undiversified Business Owner. *The Journal of Small Business Finance*, 4(2), 191-196.

Holm, H.J., S. Opper and V. Nee, 2013. Entrepreneurs Under Uncertainty: An Economic Experiment in China. *Management Science*, 59(7), 1671-1687.

Hurst, E. and B.W. Pugsley, 2011. What Do Small Businesses Do? NBER working paper 17041, DOI 10.3386/w17041, revised December 2011.

Hvide, H.K. and G. A. Panos, 2014. Risk tolerance and entrepreneurship, *Journal of Financial Economics* 111, 200-223.

Kartashova, K., 2014. Private equity premium puzzle revisited. *American Economic Review* 104, 3297-3334.

Kerins, F., Smith, J.K., Smith, R., 2004. Opportunity cost of capital for venture capital investors and entrepreneurs. *Journal of Financial and Quantitative Analysis* 39, 385-405.

Koudstaal, M., R. Sloof and M. van Praag, 2016. Risk, Uncertainty, and Entrepreneurship: Evidence from a Lab-in-the-Field Experiment, *Management Science*, 62(10), 2897-2915

Landier A. and D. Thesmar, 2009. Financial Contracting with Optimistic Entrepreneurs, *The Review of Financial Studies*, 22(1) 117-150.

Little, I.M.D., 1962. Higgledy Piggledy Growth. *Bulletin of the Oxford University Institute of Economics & Statistics* 24(4), 387-412.

Moskowitz, T.J., Vissing-Jorgensen, A., 2002. The returns to entrepreneurial investment: a private equity premium puzzle? *American Economic Review* 92, 745-778.

Mueller, E., 2011. Returns to private equity — idiosyncratic risk does matter! *Review of Finance* 15, 545-574.

Pattitoni, P., and Savioli, M. (2011). Investment choices: Indivisible non-marketable assets and suboptimal solutions. *Economic Modelling*, 28(6), 2387-2394.

Pereiro, L. E. 2010. The beta dilemma in emerging markets. *Journal of Applied Corporate Finance*, 22(5), 110-122.

Pratt, S. P., R. F. Reilly, and R. P. Schweihs, 1981. *Valuing a business: the analysis and appraisal of closely-held companies.* Dow Jones-Irwin.

Reddaway, W. B., 1967. Higgledy Piggledy Growth. *The Economic Journal* 77(307), 595-598.

Xu, H. and M. Ruef, 2004. The myth of the risk-tolerant entrepreneur. *Strategic Organization*, 2(4), 331355.

Appendix

A1. Proofs

A1.1 Proposition 1

Proof of part (a) In the equations below, the first line says that the competing portfolios, $p = (j, p_i^*)$, should receive the same mean–variance score, each calculated as $\mathrm{E}(\tilde{r}_p - r_0) - \frac{\eta_i}{2}\mathrm{var}(\tilde{r}_p)$, the mean-variance maximand or Markowitz Lagrangian. The second line immediately uses a property of efficient portfolios, $\mathrm{E}(\tilde{r}_{p_i^*} - r_0) = \eta_i\,\mathrm{var}(\tilde{r}_{p_i^*})$, to rewrite η_i as the chosen portfolio's excess-return/risk ratio. In the third line we rearrange:

$$\begin{aligned}
\mathrm{E}(\tilde{r}_j - r_0) - \frac{\eta_i}{2}\mathrm{var}(\tilde{r}_j) &= \mathrm{E}(\tilde{r}_{p_i^*} - r_0) - \frac{\eta_i}{2}\mathrm{var}(\tilde{r}_{p_i^*}); \\
\Rightarrow \mathrm{E}(\tilde{r}_j - r_0) - \frac{1}{2}\frac{\mathrm{E}(\tilde{r}_{p_i^*} - r_0)}{\mathrm{var}(\tilde{r}_{p_i^*})}\,\mathrm{var}(\tilde{r}_j) &= \frac{1}{2}\,\mathrm{E}(\tilde{r}_{p_i^*} - r_0); \\
\Rightarrow \mathrm{E}(\tilde{r}_j - r_0) &= \frac{1}{2}\left[1 + \frac{\mathrm{var}(\tilde{r}_j)}{\mathrm{var}(\tilde{r}_{p_i^*})}\right]\mathrm{E}(\tilde{r}_{p_i^*} - r_0). \qquad \text{(A.1)}
\end{aligned}$$

To obtain a CAPM-like expression we write p_i^*'s return and risk in terms of the market's return and risk: from $\tilde{r}_{p_i^*} - r_0 = w_{i,M}\,(\tilde{r}_m - r_0)$ we have $\mathrm{var}(\tilde{r}_{p_i^*}) = w_{i,M}^2\,\mathrm{var}(\tilde{r}_M)$ and $\mathrm{E}(\tilde{r}_{p_i^*} - r_0) = w_{i,M}\,\mathrm{E}(\tilde{r}_M - r_0)$. The result, after rearranging, is Equation (6). ■

Proof of part (b) The remainder of the proof continues with $\tilde{r}_{p_i^*}$, for notational compactness. We want to look behind the return-on-value version, which is about the net payoffs scaled by the PV that fits the model itself. That PV cannot be observed in an asset market, here, but we can infer it from Equation (A.1) written as

$$V_{i,0}: \quad \left[\frac{\mathrm{E}(\tilde{x})}{V_{i,0}} - 1\right] - r_0 = \frac{1}{2}\left[1 + \frac{\frac{\mathrm{var}(\tilde{x})}{V_{i,0}^2}}{\mathrm{var}(\tilde{r}_{p_i^*})}\right]\mathrm{E}(\tilde{r}_{p_i^*} - r_0). \qquad \text{(A.2)}$$

Multiplying both sides by $V_{i,0}^2$, our problem is recognised as a quadratic,

$$V_{i,0}\,\mathrm{E}(\tilde{x}) - (1 + r_0)V_{i,0}^2 = V_{i,0}^2\frac{1}{2}\mathrm{E}(\tilde{r}_{p_i^*} - r_0) + \frac{1}{2}\frac{\mathrm{var}(\tilde{x})}{\mathrm{var}(\tilde{r}_{p_i^*})}\mathrm{E}(\tilde{r}_{p_i^*} - r_0); \qquad \text{(A.3)}$$

$$\Rightarrow 0 = \underbrace{\left[(1 + r_0) + \frac{1}{2}\mathrm{E}(\tilde{r}_{p_i^*} - r_0)\right]}_{=a=:1+r_{i,f},\text{ see (7)}} V_{i,0}^2 + V_{i,0}\underbrace{[-\mathrm{E}(\tilde{x})]}_{=b} + \underbrace{\frac{1}{2}\frac{\mathrm{var}(\tilde{x})}{\mathrm{var}(\tilde{r}_{p_i^*})}\mathrm{E}(\tilde{r}_{p_i^*} - r_0)}_{=c=:\eta_i/2\,\mathrm{var}(\tilde{x})}. \qquad \text{(A.4)}$$

Thus,[25]

$$V_{i,0} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{\mathrm{E}(\tilde{x}) + \sqrt{[\mathrm{E}(\tilde{x})]^2 - 2\,(1 + r_{i,f})\,\eta_i\,\mathrm{var}(\tilde{x})}}{2\,(1 + r_{i,f})}. \quad \blacksquare \tag{A.5}$$

Proof of Corollary 1 The first version follows immediately from Equation (8). The second version writes η in terms of SR_M, to compare with *e.g.* the CAPM-based counterpart. $\blacksquare$

A1.2 Proposition 2

Proof of part (a) From Equation (17), the equal-utility condition means that the project's addition to the portfolio's expectation must balance its addition to the portfolio's risk:

$$\frac{\mathrm{E}(\tilde{z}) - V_{i,0}(1 + r_0)}{V_{i,0} + L_{i,0}} = \frac{\eta_i}{2}\frac{\mathrm{var}(\tilde{z})}{(V_{i,0} + L_{i,0})^2}, \tag{A.6}$$

$$= \frac{\eta_i}{2}\frac{V_{i,0}^2\,\mathrm{var}(\tilde{\epsilon}_j)}{(V_{i,0} + L_{i,0})^2}, \tag{A.7}$$

where in line two we use $\tilde{z}/V_{i,0} - 1 = \tilde{y}/V_{i,0} - 1 - \beta_j\,\tilde{r}_M = \tilde{r}_j - \beta_j\,\tilde{r}_M = \alpha_j + \tilde{\epsilon}_j$, with α_j and $\tilde{\epsilon}_j$ as in the market-model, $\tilde{r}_j = \alpha_j + \beta_j\,\tilde{r}_M + \tilde{\epsilon}_j$. Multiply both sides by $(V_{i,0} + L_{i,0})/V_{i,0}$ and use Equation (16). The result is

$$\frac{\mathrm{E}(\tilde{y}) - \frac{\mathrm{cov}(\tilde{y},\tilde{r}_M)}{\mathrm{var}(\tilde{r}_M)}\mathrm{E}(\tilde{r}_M - r_0) - V_{i,0}(1 + r_0)}{V_{i,0}} = \frac{\eta_i}{2}\frac{V_{i,0}\,\mathrm{var}(\tilde{\epsilon}_j)}{(V_{i,0} + L_{i,0})}. \tag{A.8}$$

Lastly, use $\tilde{y}/V_{i,0} =: 1 + \tilde{r}_j$, $V_{i,0}/(V_{i,0} + L_{i,0}) =: \omega_j$, and $\eta_i = \mathrm{E}(\tilde{r}_{p_i^*} - r_0)/\mathrm{var}(\tilde{r}_{p_i^*})$. $\blacksquare$

Proof of part (b) Multiplying both sides of Equation (A.6) by $(V_{i,0} + L_{i,0})^2$, we again obtain

[25] The relevant root is the $+\sqrt{\cdot}$ one: then the solution is somewhat below $2\mathrm{E}(\tilde{x})/[2(1 + r_{i,f})]$, while with the $-\sqrt{\cdot}$ root, the solution would be close to zero.

a quadratic equation to be solved:[26]

$$
\begin{aligned}
[\mathrm{E}(\tilde{z}) - V_{i,0}(1+r_0)]\,(V_{i,0} + L_{i,0}) &= \tfrac{\eta_i}{2}\mathrm{var}(\tilde{z}),\\
\Rightarrow \underbrace{(1+r_0)}_{=:\,a} V_{i,0}^2 + \underbrace{[-\mathrm{E}(\tilde{z}) + L_{i,0}(1+r_0)]}_{=:\,b}\, V_{i,0} + \underbrace{\frac{\eta_i}{2}\mathrm{var}(\tilde{z}) - \mathrm{E}(\tilde{z})\,L_{i,0}}_{=:\,c} &= 0. \quad \blacksquare \qquad \text{(A.9)}
\end{aligned}
$$

A.2 Deconvolution procedure: outline

The probabilities $\mathrm{pr}(\rho_i)$ that we would like to study are unobserved *per se*, but can be estimated because they link the unconditional probabilities of estimates to the conditional densities:

$$\mathrm{pr}(\widehat{\rho}_j) = \sum_{i=1}^{2I} \mathrm{pr}(\widehat{\rho}_j|\rho_i)\,\mathrm{pr}(\rho_i). \tag{A.10}$$

While we do not know $\mathrm{pr}(\widehat{\rho}_j)$, we do observe a noisy version, namely the sample estimates obtained by sorting the $\widehat{\rho}$-s from the Orbis sample into our $2J$-buckets grid for possible estimates. What we need to add, then, is the conditional probabilities $\mathrm{pr}(\widehat{\rho}_j|\rho_i)$. These are identified via a standard simulation: given ρ_i, we generate a large number (N) of time series of length 10 and thence calculate N estimates.[27] The N simulated estimates are then sorted into the grid and tabulated in the relevant row of the $\mathrm{pr}(\widehat{\rho}_j|\rho_i)$ matrix, where each of the $2I$ rows shows the conditional density of the estimates for a particular true ρ_i. If we choose a large N, each such row provides an arbitrarily close approximation to the density $\mathrm{pr}(\widehat{\rho}_j|\rho_i)$.

[26] The parameters E() and std() bear on the hedged cashflow, so solving the equation provides the value hedged. But, from the definition in (16), $\tilde{z}$ equals $\tilde{y} - [...]\,(\tilde{r}_M - r_0)$. The PV of $\tilde{r}_M - r_0$, the return on a fully levered market investment, is zero, so that the present value of the hedged project coincides with the total value.

[27] Label each simulated data series and the resulting $\widehat{\rho}$ by a superscript $n = 1, \cdots, N$. For each estimate n given true value ρ_i we start from a number, $y_0^{n,i}$, randomly drawn from a normal distribution with mean zero and variance $1 - \rho_i^2$; we then generate nine subsequent observations as $y_t^{n,i} = \rho_i\, y_{t-1}^{n,i} + e_t^{n,i}$, with $e_t^{n,i}$ a standard normal; and lastly we compute $\widehat{\rho}^{n,i}$.

In vector notation, Equation (A.10) can be written as

$$p = M' q, \tag{A.11}$$

where p denotes the $2J \times 1$ vector containing the marginal probabilities $\text{pr}(\widehat{\rho}_j)$, q the $2J \times 1$ vector containing the probabilities $\text{pr}(\rho_i)$, and M the $2I \times 2J$ matrix of conditional probabilities $\text{pr}(\widehat{\rho}_j|\rho_i)$. So $\widehat{p}$, the estimate of p, can be written as

$$\begin{aligned} \widehat{p} &= p + e, \\ &= M' q + e, \end{aligned} \tag{A.12}$$

with e a vector of $2J$ errors, which have a zero expectation and a familiar covariance matrix:

$$V := [\text{Var}(\widehat{p})] = [\text{diag}(p) - pp']. \tag{A.13}$$

Equations (A.12)-(A.13) describe a GLS-style regression problem, apart from the wrinkle that $\widehat{V}$ is singular, the sum of the estimated probabilities being identically unity in any sample. So the weighted least-squares estimator q is

$$\widehat{q} = (M'\widehat{V}^{+}M)^{-1}M'\widehat{V}^{+}\widehat{p}, \tag{A.14}$$

where $\widehat{V}$ is the estimated version of V,

$$\widehat{V} = [\text{diag}(\widehat{p}) - \widehat{p}\widehat{p}'], \tag{A.15}$$

and $\widehat{V}^{+}$ is the Moore-Penrose inverse of $\widehat{V}$. Since the elements in $\widehat{p}$ (the regressee) and in each of the $2I$ columns of M (the regressors) always sum to exactly unity, also $\widehat{q}$ must automatically sum to unity.

A3. Additional tables

The first table provides the means, highest, and lowest observations to complement the medians and standard deviations shown in Table 3. In the second table we show the CCs for sample-based growth rates, not the uniform 3% g from the main text. Lastly, we list all symbols and variables alphabetically and add their definitions.

Table A.1: **Additional descriptives**

	Total Assets			Sales		
group	mean	min	max	mean	min	max
1	81.866	3.039	921.786	96.807	3.481	1 157.283
2	226.888	40.757	1 405.768	229.533	22.793	1 666.310
3	549.765	129.093	2 587.444	533.956	44.765	2 845.570
4	1 503.540	379.614	6 009.755	1 509.157	120.720	6 445.958
5	33 959.128	1 631.195	685 941.353	32 381.956	614.866	599 152.370
	$E_0(\tilde{y}_1)/TA_0 \times 10$			growth		
group	mean	min	max	mean	min	max
1	4.201	-11.079	29.515	-0.102	-2.099	1.169
2	1.338	-1.696	6.909	-0.036	-1.903	1.119
3	0.998	-0.877	5.085	0.009	-1.805	1.083
4	0.912	-0.555	4.325	0.043	-1.593	1.051
5	0.892	-0.551	4.036	0.041	-1.526	1.017
	residual SD			correlation		
group	mean	min	max	mean	min	max
1	4.577	0.168	53.929	-0.063	-0.699	0.645
2	1.103	0.076	7.705	-0.055	-0.689	0.646
3	0.731	0.053	4.948	-0.072	-0.683	0.634
4	0.592	0.046	3.996	-0.097	-0.684	0.612
5	0.505	0.041	3.590	-0.109	-0.680	0.609

Notes The table provides the means, highest, and lowest observations to complement the medians and standard deviations shown in Table 3.

Table A.2: **Purely income-based Costs of Capital, in percent, for fully and partially committed investors**

size group	full	partial commitment						
		L=-50	L=-5	L = 0	L=5	L=50	L=100	L=1000
		covariances with $\tilde{r}_M$ as estimated; $\eta_i = 2$						
1	13.784	4.032	3.997	3.995	3.993	3.979	3.970	3.944
2	11.672	4.009	3.985	3.984	3.983	3.977	3.973	3.966
3	10.957	–	–	–	–	–	–	–
4	10.541	–	–	–	–	–	–	–
5	10.557	–	–	–	–	–	–	–
		covariances with $\tilde{r}_M$ as estimated; $\eta_i = 3$						
1	12.476	4.096	4.031	4.027	4.024	4.001	3.988	3.948
2	9.368	4.040	3.996	3.994	3.992	3.983	3.978	3.966
3	8.577	–	–	–	–	–	–	–
4	8.195	–	–	–	–	–	–	–
5	8.243	–	–	–	–	–	–	–
		covariances with $\tilde{r}_M$ set zero; $\eta_i = 2$						
1	13.784	4.115	4.069	4.066	4.064	4.047	4.037	4.008
2	11.672	4.056	4.024	4.022	4.022	4.014	4.011	4.002
3	10.957	–	–	–	–	–	–	–
4	10.541	–	–	–	–	–	–	–
5	10.557	–	–	–	–	–	–	–
		covariances with $\tilde{r}_M$ set zero; $\eta_i = 3$						
1	12.476	4.200	4.110	4.105	4.100	4.073	4.056	4.012
2	9.368	4.096	4.036	4.034	4.032	4.022	4.016	4.003
3	8.577	–	–	–	–	–	–	–
4	8.195	–	–	–	–	–	–	–
5	8.243	–	–	–	–	–	–	–

Note Data are ten years of cashflows and total assets, 2013-2022, on unlisted European firms, from Orbis. "Full" refers to full commitment: either all wealth is invested in the firm, or i at least treats it as a stand-alone asset. "Partial" commitment means the investor can borrow/lend and add optimal investments in the market index. L refers to the wealth left after the investment in the firm. We show CCs for firms that have the median characteristic in their size group. Such a 'size group' corresponds to the cell on the diagonal in a 5 by 5 double sort on the basis of total-assets and sales quintiles, 362,360 firms in total. When no convergence is reached or the proposed CC is below g, no CC is shown.

Table A.3: **List of symbols, abbreviations and variables**

symbol	definition
CC	cost of capital; to be used when valuing the operational casflow $\tilde{y}$
CE	cost of equity; to be used when valuing the cashflow after debt service, $\tilde{x}$
g	expected growth rate of the project's cashflow
i	the investor facing a non-scalable and relatively large investment project j
I_0	the project's required direct investment
j	the project
KSS	Kerins, Smith and Smith (JFQA, 2004)
$L_{i,0}$	$= W_{i,0} - I_0$, free cash available for financial investments
NPV	net present value, $V_{i,0} - I_0$
p	a portfolio
p_i^*	i's 'fall-back' portfolio, *i.e.* the preferred diversfied portfolio in the absence of the project, with weight η_M/η_i for the market index portfolio
PV	present value
$r_a, \tilde{r}_a$	return, or random return, on asset a—the project j, the risk-free asset 0, the market M, the portfolio p_i^*, etc.
$\overline{r}$	shorthand for $\mathrm{E}(\tilde{r}_j)$, the CC
RRA	relative risk aversion
SME	small/medium enterprise
SR	Sharpe Ratio for asset a, $\mathrm{E}(\tilde{r}_a - r_0)/\mathrm{std}(\tilde{r}_a)$
TA	total assets
$V_{i,0}$	the project's valuation by i, the highest subjective value that aligns the portfolios with and without the project in terms of mean–variance utility to i. Its only function is to deliver an NPV.
$w_{i,M}$	weight for the market index portfolio in i's fallback portfolio p_i^*, equal to η_M/η_i
$W_{i,0}$	investor i's initial wealth, before the NPV from the project
$\tilde{x}$	the project's time-1 payoff in the wide sense, *i.e.* operational cashflow $\tilde{y}$ minus debt service or plus return from investing any left-over cash in the market index.
$\tilde{y}$	the project's time-1 purely operational payoff (before debt service etc.)
$\tilde{Y}_1$	the project's time-1 purely operational payoff $\tilde{y}_1$ (before debt service etc.) plus the PV of subsequent $\tilde{y}$s conditional on y_1
$\tilde{z}$	the project's time-1 market-hedged payoff
β_j	the standard market-model beta from $\tilde{r}_j = \alpha_j + \beta_j\,\tilde{r}_M + \tilde{\epsilon}_j$
$\Delta_{i,0}$	the project's NPV according to i's valuation $V_{i,0}$
ϵ_j	the standard market-model residual from $\tilde{r}_j = \alpha_j + \beta_j\,\tilde{r}_M + \tilde{\epsilon}_j$
η	Relative risk aversion, RRA
ω_a	weight for risky asset a in i's optimal portfolio with the project included; a here stands for M (the market), or j (the project)